Addendum to Puzzle Corner

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In response to Puzzle Corner 67, we received an interesting solution from Associate Professor Richard Ollerton of Charles Sturt University. In particular, this contribution provides an alternative solution and technique which answers and generalises 'Problem 1' given in the following section.

Problem 'Inverting a derivative' from Puzzle Corner 67

Almost everyone knows that $dx/dy = 1/(dy/dx)$, but what about second derivatives? Does this simple reciprocal relationship persist?

Problem 1. Find the corresponding relationship between d^2x/dy^2 and d^2y/dx^2.

Problem 2. For what functions is it true that

$$\frac{d^2x}{dy^2} = \left(\frac{d^2y}{dx^2}\right)^{-1}?$$

Generalisation and Alternative Solution to 'Problem 1'

We provide here an edited version of Richard's submission as follows.

The solution to Puzzle Corner's 'Inverting a derivation' problem extends the well-known first derivative formula $(dx/dy)(dy/dx) = 1$ to second derivatives to obtain

$$\frac{d^2y}{dx^2} = -\left(\frac{dy}{dx}\right)^3 \frac{d^2x}{dy^2}.$$

Naturally (as mathematicians), we also wonder about similar formulas for higher derivatives. The direct approach of successive differentiation of the second derivative and resulting higher order formulas leads quickly to algebraic mires of product rules and chain rules. However, the following approach provides a more tractable way of generalising the result above. Existence and non-zero derivative conditions should be assumed as required.

For integers $m, n \geq 0$, let $\delta_{m,n} = 1$ if $m = n = 0$, and $\delta_{m,n} = 0$ otherwise. Then from the chain rule and the general Leibniz rule, we observe that

$$\delta_{m,0} = \frac{d^{m+1}x}{dx^{m+1}} = \frac{d^m}{dx^m}\left(\frac{dx}{dy}\frac{dy}{dx}\right) = \sum_{j=0}^{m} \binom{m}{j} \left(\frac{d^j}{dx^j}\frac{dx}{dy}\right) \frac{d^{m-j+1}y}{dx^{m-j+1}}. \qquad (1)$$

We may then rearrange equation (1) to solve for the last term of the summation as

$$\left(\frac{d^m}{dx^m}\frac{dx}{dy}\right)\frac{dy}{dx} = \delta_{m,0} - \sum_{j=0}^{m-1} \binom{m}{j} \left(\frac{d^j}{dx^j}\frac{dx}{dy}\right) \frac{d^{m-j+1}y}{dx^{m-j+1}}.$$

Notice here that we use the convention that a sum evaluates to zero if its upper limit is less than its lower limit. In particular, this means that the sum above is zero if $m = 0$.

Moreover, from the chain and the general Leibniz rules, we have

$$\begin{aligned}\frac{d^{m+1}}{dx^{m+1}}\frac{d^n x}{dy^n} = \frac{d^m}{dx^m}\left(\frac{dy}{dx}\frac{d}{dy}\frac{d^n x}{dy^n}\right) &= \frac{d^m}{dx^m}\left(\frac{d^{n+1}x}{dy^{n+1}}\frac{dy}{dx}\right)\\ &= \sum_{j=0}^{m}\binom{m}{j}\left(\frac{d^j}{dx^j}\frac{d^{n+1}x}{dy^{n+1}}\right)\frac{d^{m-j+1}y}{dx^{m-j+1}}.\end{aligned}$$

Solving for the last term in the summation, we find that

$$\left(\frac{d^m}{dx^m}\frac{d^{n+1}x}{dy^{n+1}}\right)\frac{dy}{dx} = \frac{d^{m+1}}{dx^{m+1}}\frac{d^n x}{dy^n} - \sum_{j=0}^{m-1}\binom{m}{j}\left(\frac{d^j}{dx^j}\frac{d^{n+1}x}{dy^{n+1}}\right)\frac{d^{m-j+1}y}{dx^{m-j+1}}.$$

After an induction, for each integer $s \geq n \geq 0$, we see that

$$\begin{aligned}&\left(\frac{d^{s-n}}{dx^{s-n}}\frac{d^{n+1}x}{dy^{n+1}}\right)\left(\frac{dy}{dx}\right)^{n+1}\\ &\qquad = \delta_{s,0} - \sum_{k=0}^{n}\sum_{j=0}^{s-k-1}\binom{s-k}{j}\left(\frac{d^j}{dx^j}\frac{d^{k+1}x}{dy^{k+1}}\right)\frac{d^{s-j-k+1}y}{dx^{s-j-k+1}}\left(\frac{dy}{dx}\right)^k.\end{aligned}$$

If we then let $m = s - n \geq 0$, we obtain the equality

$$\begin{aligned}&\left(\frac{d^m}{dx^m}\frac{d^{n+1}x}{dy^{n+1}}\right)\left(\frac{dy}{dx}\right)^{n+1}\\ &= \delta_{m,n} - \sum_{k=0}^{n}\sum_{j=0}^{m+n-k-1}\binom{m+n-k}{j}\left(\frac{d^j}{dx^j}\frac{d^{k+1}}{dy^{k+1}}\right)\frac{d^{m+n-j-k+1}y}{dx^{m+n-j-k+1}}\left(\frac{dy}{dx}\right)^k \qquad (2)\end{aligned}$$

for each integer $m, n \geq 0$.

Equation (2) can then be viewed as a linear partial recurrence relation where each coefficient is a product of a binomial coefficient and an x-derivative of y, namely,

$$f(m,n)\frac{dy}{dx} = \delta_{m,n} - \sum_{k=0}^{n}\sum_{j=0}^{m+n-k-1}\binom{m+n-k}{j}\frac{d^{m+n-j-k+1}y}{dx^{m+n-j-k+1}}f(j,k)$$

where

$$f(m,n) = \left(\frac{d^m}{dx^m}\frac{d^{n+1}x}{dy^{n+1}}\right)\left(\frac{dy}{dx}\right)^n.$$

Now, considering the case where $m = 0$, we see that

$$f(0,0) = \frac{dx}{dy}, \qquad f(0,n) = \frac{d^{n+1}x}{dy^{n+1}}\left(\frac{dy}{dx}\right)^n$$

and

$$f(0,n)\frac{dy}{dx} = \delta_{0,n} - \sum_{k=0}^{n-1}\sum_{j=0}^{n-k-1}\binom{n-k}{j}\frac{d^{n-j-k+1}y}{dx^{n-j-k+1}}f(j,k). \qquad (3)$$

Notice here that the limit of the outer sum in equation (3) is $n-1$ as otherwise the n-th term would be zero. From equation (3), we find that

$$\begin{aligned} f(0,0)\frac{dy}{dx} &= \frac{dx}{dy}\frac{dy}{dx} = 1 \\ f(0,1)\frac{dy}{dx} &= -\binom{1}{0}\frac{d^2y}{dx^2}f(0,0) = -\frac{d^2y}{dx^2}\frac{dx}{dy} \\ f(0,2)\frac{dy}{dx} &= -\binom{2}{0}\frac{d^3y}{dx^2}f(0,0) - \binom{2}{1}\frac{d^2y}{dx^2}f(1,0) - \binom{1}{0}\frac{d^2y}{dx^2}f(0,1) \\ &= -\frac{d^3y}{dx^3}\frac{dx}{dy} + 3\left(\frac{d^2y}{dx^2}\right)^2\left(\frac{dx}{dy}\right)^2 \end{aligned}$$

since

$$f(1,0)\frac{dy}{dx} = -\binom{1}{0}\frac{d^2y}{dx^2}f(0,0) = -\frac{d^2y}{dx^2}\frac{dx}{dy}$$

and thus,

$$f(1,0) = -\frac{d^2y}{dx^2}\left(\frac{dx}{dy}\right)^2.$$

Applying the definition of $f(0,n)$ above and collecting these results for closer inspection, we have

$$\begin{aligned} \frac{dx}{dy}\frac{dy}{dx} &= 1 \\ \frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^2 + \frac{d^2y}{dx^2}\frac{dx}{dy} &= 0 \\ \frac{d^3x}{dy^3}\left(\frac{dy}{dx}\right)^3 + \frac{d^3y}{dx^3}\frac{dx}{dy} &= 3\left(\frac{d^2y}{dx^2}\right)^2\left(\frac{dx}{dy}\right)^2 \end{aligned}$$

where the second result is the original solution to the Puzzle Corner problem. These equations suggest patterns among the terms of the left-hand sides. We may obtain similar results for higher order terms as follows.

Notice that in the summation in equation (3), the $d^{n+1}y/dx^{n+1}$ term arises only when $(j,k) = (0,0)$. Consequently, for each $n \geq 1$, we may rewrite equation (3) as

$$\frac{d^{n+1}x}{dy^{n+1}}\left(\frac{dy}{dx}\right)^{n+1} + \frac{d^{n+1}y}{dx^{n+1}}\frac{dx}{dy} = -\sum_{k=0}^{n-1}\sum_{\substack{j=0 \\ (j,k)\neq(0,0)}}^{n-k-1}\binom{n-k}{j}\frac{d^{n-j-k+1}y}{dx^{n-j-k+1}}f(j,k) \qquad (4)$$

where the summation includes derivatives of order at most n. Thus, equation (4) provides an algebraic relationship between $d^{n+1}x/dy^{n+1}$ and $d^{n+1}y/dx^{n+1}$ in terms of lower order derivatives and generalises our familiar friend $(dx/dy)(dy/dx) = 1$.

But now we wonder if there is a version of equation (4) that produces a symmetrical form for each result. The left-hand side of equation (4) can be made symmetrical in x and y by dividing by $(dy/dx)^{n/2}$. The corresponding right-hand side is then equal to itself with x and y interchanged. Thus, we find two equivalent right-hand sides by interchanging x and y. We set the symmetric left-hand side equal to the average of these two right-hand sides to obtain a symmetric version of equation (4). To illustrate, consider equation (4) with $n = 2$, that is,

$$\frac{d^3x}{dy^3}\left(\frac{dy}{dx}\right)^3 + \frac{d^3y}{dx^3}\frac{dx}{dy} = 3\left(\frac{d^2y}{dx^2}\right)^2\left(\frac{dx}{dy}\right)^2.$$

Dividing both sides by dy/dx, we obtain

$$\frac{d^3x}{dy^3}\left(\frac{dy}{dx}\right)^2 + \frac{d^3y}{dx^3}\left(\frac{dx}{dy}\right)^2 = 3\left(\frac{d^2y}{dx^2}\right)^2\left(\frac{dx}{dy}\right)^3 = 3\left(\frac{d^2x}{dy^2}\right)^2\left(\frac{dy}{dx}\right)^3$$

and thus

$$\frac{d^3x}{dy^3}\left(\frac{dy}{dx}\right)^2 + \frac{d^3y}{dx^3}\left(\frac{dx}{dy}\right)^2 = \frac{3}{2}\left(\left(\frac{d^2y}{dx^2}\right)^2\left(\frac{dx}{dy}\right)^3 + \left(\frac{d^2x}{dy^2}\right)^2\left(\frac{dy}{dx}\right)^3\right)$$

which is symmetrical in x and y. This result can then be simplified, while maintaining symmetry, by incorporating the second order result to give

$$\frac{d^3x}{dy^3}\left(\frac{dy}{dx}\right)^2 + \frac{d^3y}{dx^3}\left(\frac{dx}{dy}\right)^2 = -3\frac{d^2x}{dy^2}\frac{d^2y}{dx^2}.$$

This final result is itself something of a generalisation of $(dx/dy)(dy/dx) = 1$ to products of second derivatives, and could be considered to be an alternative solution of the original Puzzle Corner problem which calls for a "corresponding relationship" between the second derivatives.

Naturally, we also wonder about generalising this result to products of third and higher order derivatives, and invite the interested reader to investigate.

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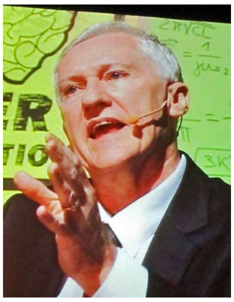

Peter Higgins is a Professor of Mathematics at the University of Essex. He is the inventor of Circular Sudoku, a puzzle type that has featured in many newspapers, magazines, books, and computer games all over the world. He has written extensively on the subject of mathematics and won the 2013 Premio Peano Prize in Turin for the best book published about mathematics in Italian in 2012. Originally from Australia, Peter has lived in Colchester, England with his wife and four children since 1990.