NCCA
An Chomhairle Náisiúnta Curaclaim agus Measúnachta
National Council for Curriculum and Assessment

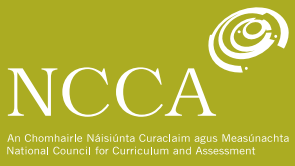
Research Report No. 17

Mathematics in Early Childhood and Primary Education (3–8 years)

Definitions, Theories, Development and Progression

Elizabeth Dunphy, Thérèse Dooley and Gerry Shiel

With Deirdre Butler, Dolores Corcoran, Miriam Ryan and Joe Travers
International Advisor: Professor Bob Perry



© NCCA 2014
ISSN 1649-3362

National Council for Curriculum and Assessment
35 Fitzwilliam Square, Dublin 2.

T: +353 1 661 7177
F: +353 1 661 7180
E: info@ncca.ie

www.ncca.ie

Mathematics in Early Childhood and Primary Education (3–8 years)

Definitions, Theories, Development and Progression

Elizabeth Dunphy, St. Patrick's College, Dublin

Thérèse Dooley, St. Patrick's College, Dublin

Gerry Shiel, Educational Research Centre, Dublin

Deirdre Butler, St. Patrick's College, Dublin

Dolores Corcoran, St. Patrick's College, Dublin

Miriam Ryan, St. Patrick's College, Dublin

Joe Travers, St. Patrick's College, Dublin

Professor Bob Perry, Charles Sturt University, Australia

Research conducted on behalf of the
National Council for Curriculum and Assessment

2014

Acronyms

AAMT	Australian Association of Mathematics Teachers
Aistear	The Early Childhood Curriculum Framework (2009)
CCSSM	Common Core States Standards for Mathematics (United States)
CHAT	Cultural historical activity theory
DEIS	Delivering Equality of Opportunities in Schools
DES	Department of Education and Skills (formerly Department of Education and Science)
DfEE	Department for Education and Employment (United Kingdom)
EAL	English as an Additional Language
ECA	Early Childhood Australia
ENRP	Early Numeracy Research Project (Victoria, Australia)
ERC	Educational Research Centre
HLT	Hypothetical Learning Trajectory
ICT	Information and Communication Technology
KDU	Key Developmental Understanding
LFIN	Learning Framework in Number (Wright, Martland & Stafford, 2006)
LT	Learning Trajectory
NAEYC	National Association for the Education of Young Children (United States)
NCCA	National Council for Curriculum and Assessment
NCTM	National Council of Teachers of Mathematics (United States)
NGA	National Governors Association (United States)
NRC	National Research Council (United States)
OECD	Organisation for Economic Cooperation and Development
PISA	Programme for International Student Assessment
PM	Project Maths
PSC	Primary School Curriculum (1999)
PSMC	Primary School Mathematics Curriculum (1999)
RME	Realistic Mathematics Education
RTI	Response to Intervention (United States Initiative)
STEM	Science, Technology, Engineering and Mathematics
TAL	Tussendoelen Annex Leerlijnen (in Dutch); Intermediate Attainment Targets (in English)
TIMSS	Trends in International Mathematics and Science Study

Table of Contents

Executive Summary 7

A View of Mathematics 8

Context 8

Definitions of Mathematics Education 9

Theoretical Perspectives 9

Language and Communication 10

Defining Goals 10

The Development of Children's Mathematical Thinking 11

Assessing and Planning for Progression 11

Addressing Diversity 12

Key Implications 13

A View of Mathematics 15

Introduction 19

Context 20

Developing Mathematics Education in Ireland for Children Aged 3–8 Years 20

Curriculum Context 22

Performance Context 23

Policy Context 24

Linguistic and Social Contexts 27

Overview of Chapters 28

Chapter 1: Defining Mathematics Education 31

The Foundations of Mathematics 32

A Definition of Mathematics Education 33

Numeracy 34

Defining Mathematics Education for Children Aged 3–8 Years 36

A Key Aim of Mathematics Education: Mathematical Proficiency 37

Mathematization 38

Conclusion 40

Chapter 2: Theoretical Perspectives 41

Sociocultural Perspectives 43
A Cultural-Historical Activity Theory Perspective 45
A Situative Perspective 46
Cognitive Perspectives 47
Constructivist Perspectives 47
Constructionism 48
A Redeveloped Primary School Mathematics Curriculum 51
Implications for Practice 52
Conclusion 56

Chapter 3: Language, Communication and Mathematics 57

The Role of Language in Developing Mathematical Knowledge 59
Adult Support 60
The Nature and Scope of Mathematical Discourse 61
Establishing a Math-Talk Culture 63
Learning Mathematical Vocabulary 64
Variation in Language Skills and Impact on Mathematics 65
Conclusion 66

Chapter 4: Defining Goals 69

A Coherent Curriculum 70
Specifying Goals 71
Overarching Goals for Mathematics Education 71
Higher-Order Thinking 71
Engaging with Powerful Mathematical Ideas 72
Exploring the Big Ideas in Mathematics Learning 72
The Structure of Curriculum Materials 74
Breaking Down the Goals: Critical Transitions within Mathematical Domains 75
Conclusion 78

Chapter 5: The Development of Children's Mathematical Thinking 79

A Historical Perspective 80
From Stages of Development to Levels of Sophistication in Thinking 81

Developing Children's Mathematical Thinking: Three Approaches 81
The First Approach: Working with Children's Thinking and Understanding (RME) 82
Key Features 82
The Teacher's Task 83
The Second Approach: Teacher-Generated Hypothetical Learning Trajectories (Simon) 85
The Third Approach: Pre-Specified Developmental Progressions as a Basis for Learning Trajectories (Sarama and Clements) 87
Comparing the Three Approaches 90
Definitions and Characteristics 90
Recognising Diverse Routes in Learning 92
Recognising Developmental Variation 93
Curriculum Development and the Role of Learning Trajectories 94
Supporting Teachers in Planning 94
Supporting Learning for Pre-Service Teachers 95
Conclusion 96

Chapter 6: Assessing and Planning for Progression 97

Assessing Mathematics Learning in Early Childhood 98
Formative Assessment 99
Conceptual Frameworks 99
Methods 100
Observations 100
Tasks 101
Interviews 102
Conversations 103
Pedagogical Documentation 104
Supporting Children's Progression with Formative Assessment 104
Diagnostic and Summative Assessment 106
Screening/Diagnostic Tools 108
Standardised Norm-Referenced Tests 109
Planning for Progression Using Assessment Outcomes 110
Immersion Settings 111
Children with Special Needs 112
Conclusion 113

Chapter 7: Addressing Diversity 115

Identification of Learning Difficulties in Mathematics 116
Exceptional Children 117
Intellectual and Developmental Difficulties 117
Specific Difficulties in Mathematics 118
Developmental Delay 119
Mathematically Talented Children 120
Cultural Variation 121
Ethnicity 121
Socioeconomic Disadvantage 122
Conclusion 124

Chapter 8: Key Implications 127

Glossary 131

References 137

The executive summaries of reports No. 17 and No. 18 are available online at *ncca.ie/primarymaths*. The online versions include some hyperlinks which appear as text on dotted lines in this print copy.

Acknowledgements

The authors thank the National Council for Curriculum and Assessment for commissioning and supporting this report. We are very thankful to Arlene Forster and Aoife Kelly of the NCCA for providing detailed feedback on earlier drafts of the report. The authors are also indebted to Professor Bob Perry, Charles Sturt University, Australia who read early drafts of the report and who provided expert advice on various issues addressed in the report.

Executive Summary

The review of research on mathematics learning of children aged 3–8 years is presented in two reports. These are part of the NCCA's Research Report Series (ISSN 1649–3362). The first report (Research Report No. 17) focuses on theoretical aspects underpinning the development of mathematics education for young children. The second report (Research Report No. 18) is concerned with related pedagogical implications. The key messages from Report No. 17 are presented in this Executive Summary.

A View of Mathematics

Both reports are underpinned by a view of mathematics espoused by Hersh (1997). That is, mathematics as 'a human activity, a social phenomenon, part of human culture, historically evolved, and intelligible only in a social context' (p. xi). Mathematics is viewed not only as useful and as a way of thinking, seeing and organising the world, but also as aesthetic and worthy of pursuit in its own right (Zevenbergen, Dole, & Wright, 2004). All children are viewed as having an ability to solve mathematical problems, make sense of the world using mathematics, and communicate their mathematical thinking.

Context

The context in which this report is presented is one in which there is a growing awareness of the importance of mathematics in the lives of individuals, in the economy and in society more generally. In parallel with this there is a growing realisation of the importance of the early childhood years as a time when children engage with many aspects of mathematics, both at home and in educational settings (Ginsburg & Seo, 1999; Perry & Dockett, 2008). Provision for early childhood education in Ireland has also increased. A recent development is free preschool education for all children in the year prior to school entry. In addition, a new curriculum framework, *Aistear* (National Council for Curriculum and Assessment [NCCA], 2009a; 2009b), is available to support adults in developing children's learning from birth to six years. At the same time, however, there are concerns about the levels of mathematical reasoning and problem-solving amongst school-going children, as evidenced in recent national and international assessments and evaluations at primary and post-primary levels (e.g., Eivers et al., 2010; Perkins, Cosgrove, Moran & Shiel, 2012; Jeffes et al., 2012). While the 1999 Primary School Mathematics Curriculum (PSMC) has been well received by teachers (NCCA, 2005), the Inspectorate of the then Department of Education and Science identified some difficulties with specific aspects of implementation (DES, 2005). The current report envisions a revised PSMC that is responsive to these concerns, that recognises the importance of building on children's early

engagement with mathematics, and which takes account of the changing demographic profile of many educational settings, and the increased diversity among young children.

Definitions of Mathematics Education

Current views of mathematics education are inextricably linked with ideas about equity and access and with the vision that mathematics is for all (Bishop & Forganz, 2007), i.e. all children should have opportunities to engage with and benefit from mathematics education and no child should be excluded.

Mathematics education is seen as comprising a number of mathematical practices that are negotiated by the learner and teacher within broader social, political and cultural contexts (Valero, 2009). An interpretation of mathematics that includes numeracy but is broader should underpin efforts towards curricular reform in Ireland. This report identifies mathematical proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition) (NRC, 2001) as a key aim of mathematics education. It is promoted through engagement with processes such as connecting, communicating, reasoning, argumentation, justifying, representing, problem-solving and generalising. All of these are encompassed in the overarching concept of mathematization. This involves children interpreting and expressing their everyday experiences in mathematical form and analysing real world problems in a mathematical way through engaging in these key processes (Ginsburg, 2009a; Treffers & Beishuizen, 1999). Thus mathematization is identified as a key focus of mathematics education and as such it is given considerable attention in this report. Mathematics education should address the range of mathematical ideas that all children need to engage with. It should not be limited to number.

Theoretical Perspectives

Cognitive and sociocultural perspectives provide different lenses with which to view mathematics learning and the pedagogy that can support it (Cobb, 2007). Cognitive perspectives are helpful in focusing on individual learners while sociocultural perspectives are appropriate when focusing on, for example, pedagogy (Cobb & Yackel, 1996). Sociocultural, cognitive perspectives and constructionism all offer insights which can enrich our understanding of issues related to the revision of the curriculum. They do so by providing key pointers to each of the elements of learning, teaching, curriculum and assessment. Used together they can help in envisaging a new iteration of the PSMC.

In this report, learning mathematics is presented as an active process which involves meaning making, the development of understanding, the ability to participate in increasingly skilled ways in mathematically-related activities and the development of a mathematical identity (Von Glasersfeld, 1984; Rogoff, 1998; Lave & Wenger, 1991). Learning also involves the effective use of key tools such as language, symbols, materials and images. It is seen to be supported by participation in the community of learners engaged in mathematization, in small-group and whole class conversations.

The proactive role of the teacher must be seen to involve the creation of a zone of proximal development, the provision of scaffolding for learning and the co-construction of meaning with the child based on awareness and understanding of the child's perspective (e.g., Bruner, 1996). It also involves a dialogical pedagogy of argumentation and discussion designed to support effective conceptual learning and the ability for teachers to act contingently (e.g., Corcoran, 2012).

Language and Communication

Cognitive/constructivist and sociocultural perspectives on learning emphasise the key role of language in supporting young children's mathematical development. Emerging learning theories point to the importance of mathematical discourse as a tool to learn mathematics (e.g., Sfard, 2007). In addition to introducing young children to mathematical vocabulary, it is important to engage them in 'math talk' – conversations about their mathematical thinking and reasoning (Hufferd-Ackles, Fuson & Sherin, 2004). Such talk should occur across a broad range of contexts, including unplanned and planned mathematics activities and activities such as storytelling or shared reading, where mathematics may be secondary. Children at risk of mathematical difficulties, including those living in disadvantaged circumstances, may need additional, intensive support to develop language and the ability to participate in mathematical discourse (Neuman, Newman & Dwyer, 2011).

Research indicates an association between the quality and frequency of mathematical language used by carers, parents and teachers as they interact with young children, and children's development in important aspects of mathematics (Klibanoff et al., 2006; Gentner, 2003; Levine et al., 2012). This highlights the importance of adults modelling mathematical language and encouraging young children to use such language. Conversations amongst children about mathematical ideas are also important for mathematical development (e.g., NRC, 2009).

Defining Goals

The goal statements of a curriculum should be aligned with its underlying theory. Curriculum goals should reflect new emphases on ways to develop children's mathematical understandings and to foster their identities as mathematicians (Perry & Dockett, 2002; 2008). This report proposes that processes and content should be clearly articulated as related goals (e.g., mathematization can be regarded as both a process and as content since as children engage in processes e.g., connecting, they construct new and/or deeper understandings of content). This contrasts with the design of the Primary School Mathematics Curriculum (PSMC), where content and processes are presented separately, and content is emphasised over processes. An approach in which processes are foregrounded, but content areas are also specified, is consistent with a participatory approach to mathematics learning and development.

General goals need to be broken down for planning, teaching and assessment purposes. This can be done through identifying critical ideas i.e., the shifts in mathematical reasoning required for the development of mathematical concepts (e.g., Simon, 2006; Sarama & Clements, 2009). An understanding of this framework enables teachers to provide support for children's progression towards curriculum goals.

The Development of Children's Mathematical Thinking

The idea of stages of development in children's mathematical learning (most often associated with Piaget) has now been replaced with ideas about developmental/learning paths. This is a relatively recent area of research in mathematics education (Daro et al., 2011) and as such is still under development. Learning paths are also referred to as learning trajectories. They indicate the sequences that apply in a general sense to development in the various domains of mathematics (e.g., Fosnot & Dolk, 2001; Sarama & Clements, 2009; van den Heuvel-Panhuizen, 2008). This report envisages that general learning paths will provide teachers with a basis for assessing and interpreting the mathematical development in their own classroom contexts, and will lead to learning experiences matched to individual children's needs.

There is variation in the explication of learning paths, for example, linear/nonlinear presentation, level of detail specified, mapping of paths to age/grade, and role of teaching. Different presentations reflect different theoretical perspectives. An approach to the specification of learning paths that is consistent with sociocultural perspectives is one which recognises the paths as

i. provisional, as many children develop concepts along different paths and there can never be certainty about the exact learning path that individual children will follow as they develop concepts

ii. not linked to age, since this suggests a normative view of mathematics learning

iii. emerging from engagement in mathematical-rich activity with children reasoning in, and contributing to, the learning/teaching situation (e.g., Fosnot & Dolk, 2001; Stigler & Thompson, 2012; Wager & Carpenter, 2012).

Assessing and Planning for Progression

Of the assessment approaches available, formative assessment offers most promise for generating a rich picture of young children's mathematical learning (e.g., NCCA, 2009b; Carr & Lee, 2012). Strong conceptual frameworks are important for supporting teachers' formative assessments (Carr & Lee, 2012; Ginsburg, 2009a; Sarama & Clements, 2009). These influence what teachers recognise as significant learning, what they take note of and what aspects of children's activity they give feedback on. There is a range of methods (observation, tasks, interviews, conversations,

pedagogical documentation) that can be used by educators to assess and document children's mathematics learning and their growing identities as mathematicians. Digital technologies offer particular potential in this regard. These methods are challenging to implement and require teachers to adopt particular, and for some, new, perspectives on mathematics, mathematics learning and assessment. Constructing assessments which enlist children's agency (for example, selecting pieces for inclusion in a portfolio or choosing particular digital images to tell a learning story) has many benefits. One benefit is the potential for the inclusion of children's perspectives on their learning (Perry & Dockett, 2008).

In the main, the current literature affords scant support for the use of standardised tests with children in the age range 3–8 years (e.g., Mueller, 2011). More structured teacher-initiated approaches and the use of assessment within a diagnostic framework may be required on some occasions, for example, when children are at risk of mathematical difficulties. However, research indicates a range of factors problematising the use of standardised measures with young children (e.g., Snow & Van Hemel, 2008).

The complex variety of language backgrounds of a significant minority of young children presents a challenge in the learning, teaching and assessment of mathematics. Children for whom the language of the home is different to that of the school need particular support. That support should focus on developing language, both general and mathematical, to maximise their opportunities for mathematical development and their meaningful participation in assessment (Tabors, 2008; Wood & Coltman, 1998). Educators carrying out assessment procedures such as interviews, observations or tasks in an immersion context have the dual purpose of assessing and evaluating both the mathematical competences and language competences of the child, to gain a full picture. Dual language assessment is particularly desirable in this context (Murphy & Travers, 2012; Rogers, Lin & Rinaldi, 2011).

Addressing Diversity

Mathematics 'for all' implies a pedagogy that is culturally sensitive and takes account of individuals' ways of interpreting and making sense of mathematics (Malloy, 1999; Fiore, 2012). An issue of concern is the limitations of norms-based testing which can disadvantage certain groups. This indicates the need to use a diverse range of assessment procedures to identify those who are experiencing learning difficulties in mathematics.

The groups of individuals that often require particular attention in the teaching and learning of mathematics are 'exceptional' children (those with developmental disabilities or who are especially talented at mathematics) (Kirk, Gallagher, Coleman, & Anastasiow, 2012). These individuals do not require distinctive teaching approaches, but there is a need to address their individual needs. In particular, the use of multi-tiered tasks in which different levels of challenge are incorporated is advocated (Fiore, 2012).

In addition, this report identifies the need to provide parents and educators with particular supports to ensure a mathematically-interactive and rich environment for children aged 3–8 years. It also indicates that the intensity of the support needs to vary according to the needs of particular groups of children (e.g., Ehrlich, Levine, & Goldin-Meadow, 2006).

Key Implications

The following are the key implications that arise from this report for the development of the mathematics curriculum for children aged 3–8 years:

- In the curriculum, a view of all children as having the capacity to engage with deep and challenging mathematical ideas and processes from birth should be presented. From this perspective, and in order to address on-going concerns about mathematics at school level, a curriculum for 3–8 year-old children is critical. This curriculum needs to take account of the different educational settings that children experience during these years.
- The curriculum should be developed on the basis of conversations amongst all educators, including those involved in the NCCA's consultative structures and processes, about the nature of mathematics and what it means for young children to engage in doing mathematics. These conversations should be informed by current research, as synthesised in this report and in Report No. 18, which presents a view of mathematics as a human activity that develops in response to everyday problems.
- The overall aim of the curriculum should be the development of mathematical proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition). As mathematization plays a central role in developing proficiency, the processes of mathematization should permeate all learning and teaching activities. These include connecting, communicating, reasoning, argumentation, justifying, representing, problem-solving and generalising. (*Chapter 1*)
- The curriculum should foreground mathematics learning and development as being dependent on children's active participation in social and cultural experiences, while also recognising the role of internal processes. This perspective on learning provides a powerful theoretical framework for mathematics education for young children. Such a framework requires careful explication in the curriculum and its implications for pedagogy should be clearly communicated. (*Chapter 2*)
- In line with the theoretical framework underpinning the curriculum, mathematical discourse (math talk) should be integral to the learning and teaching process. The curriculum should also promote the development of children's mathematical language in learning situations where mathematics development may not be the primary goal. Particular attention should be given to providing intensive language support, including mathematical language, to children at risk of mathematical difficulties. (*Chapter 3*)

- The goal statements of the curriculum should be aligned with its underlying theory. An approach whereby processes are foregrounded but content areas are also specified is consistent with a participatory approach to mathematics learning and development. In the curriculum, general goals need to be broken down for planning, teaching and assessment purposes. Critical ideas indicating the shifts in mathematical reasoning required for the development of key concepts should be identified. (*Chapter 4*)
- Based on the research which indicates that teachers' understanding of developmental progressions (learning paths) can help them with planning, educators should have access to information on general learning paths for the different domains. Any specification of learning paths should be consistent with sociocultural perspectives, which recognise the paths as provisional, non-linear, not age-related and strongly connected to children's engagement in mathematically-rich activity. Account needs to be taken of this in curriculum materials. Particular attention should be given to the provision of examples of practice, which can facilitate children's progression in mathematical thinking. (*Chapter 5*)
- The curriculum should foreground formative assessment as the main approach for assessing young children's mathematical learning, with particular emphasis on children's exercise of agency and their growing identities as mathematicians. Digital technologies offer particular potential in relation to these aspects of development. The appropriate use of screening/diagnostic tests should be emphasised as should the limitations of the use of standardised tests with young children. The curriculum should recognise the complex variety of language backgrounds of a significant minority of young children and should seek to maximise their meaningful participation in assessment. (*Chapter 6*)
- A key tenet of the curriculum should be the principle of 'mathematics for all'. Central to this is the vision of a multicultural curriculum which values the many ways in which children make sense of mathematics. While there are some groups or individuals who need particular supports in order to enhance their engagement with mathematics, in general distinct curricula should not be advocated. (*Chapter 7*)
- Curriculum developments of the nature described above are strongly contingent on concomitant developments in pre-service and in-service education for educators at preschool and primary levels.

A View of Mathematics

This report is concerned with definitions, theories, stages of developments and progression in mathematics in early childhood and primary education for children aged 3–8 years. It is premised on a view of mathematics as not only useful and as a way of thinking, seeing and organising the world, but also as aesthetic and worthy of pursuit in its own right (Zevenbergen, Dole, & Wright, 2004).

Mathematics is intrinsic to our comprehension of the world. Stewart (1996) gives an overview of the many patterns that are found in nature and refers, in particular, to the pattern of number (e.g., the Fibonacci numbers and petals of flowers), the patterns of form (e.g., those found in sand dunes) and the pattern of movement (e.g., the regular rhythm of the human walk). He maintains that mathematics helps us to understand nature:

> *Each of nature's patterns is a puzzle, nearly always a deep one. Mathematics is brilliant at helping us to solve puzzles. It is a more or less systematic way of digging out the rules and structures that lie behind some observed pattern or regularity, and then using those rules and structures to explain what's going on. Indeed, mathematics has developed alongside our understanding of nature, each reinforcing the other.* (p. 16)

Appreciation of all of these facets of mathematics greatly enhances children's capacities to engage fully with the world around them.

Mathematics also has a utilitarian aspect. Struik (1987) describes how, as far back as the Old Stone Age, there was a need to measure length, volume and time. Nowadays, the availability of increasingly sophisticated tools allows ever-more accurate measurements of a myriad of attributes to be obtained. Wheeler and Wheeler (1979) suggest that mathematics is a language:

> *Mathematics is the language of those who wish to express ideas of shape, quantity, size and order. It is the language that is used to describe our growing understanding of the physical universe, to facilitate the transactions of the market place, and to analyze and understand the complexities of modern society. Thus, to communicate effectively, it is essential to have a knowledge of the language.* (p. 3)

Others talk about the beauty and joy of mathematics. For example, Poincaré's 'Aha' moment (the discovery of a new expression for Fuchsian functions) as he stepped on a bus is often cited to illustrate the stages of the creative process (e.g., Hadamard, 1945; Koestler, 1969). In interviews conducted with 70 mathematicians about their work, Burton (2004) found that the majority of her participants identified something which they termed intuition, insight, or, in a few cases, instinct as

a key factor in coming to know mathematics – this insight was linked with a sense of joy. Dreyfus and Eisenberg (1986) suggest that just as individuals come to appreciate music, art and literature by understanding their underlying structures, so too they can appreciate mathematics.

However, mathematics is also linked with power. Since mathematics is behind most of society's inventions (not all for the common good!), it tends to give those who succeed in it access to wealth and power. It thus acts as a 'gatekeeper' – studies around the world show that gender, ethnicity and social class can impact on successful performance in mathematics and thus a large part of the world's population is denied access to its 'power' (e.g., Ernest, Greer, & Sriraman, 2009; Secada, 1995). While power and wealth may not seem to be of immediate concern to 3–8 year-old children, the foundations of mathematical proficiency are established during these years. Different conceptualisations of what it is to do mathematics can ameliorate such inequities and this is given attention throughout this report.

In the words of Hersh (1997, p. xi), 'mathematics is a human activity, a social phenomenon, part of human culture, historically evolved, and intelligible only in a social context'. Thus, this report and the accompanying one (Research Report No. 18) are also founded on a view of all individuals having an innate ability to solve problems and make sense of the world through mathematics.

Introduction

In this introduction we describe the broad context in which the development of a revised mathematics curriculum for children in the 3–8 year age range is embedded. This includes a description of current provision of early childhood education in Ireland. It also includes consideration of the existing Primary School Mathematics Curriculum (PSMC) and issues around its implementation, a review of performance on national and international assessments of mathematics, and an overview of recent policy initiatives related to mathematics education. Following this we look at the evolving language context in which mathematics education is provided in Irish schools and we acknowledge the range of social issues that can impact on children's mathematics learning in early education settings. We conclude with an overview of the remaining chapters in the report.

Context

The profile of mathematics as a curriculum area has increased greatly in recent years as countries seek to establish 'knowledge-based' or 'smart' economies, where many positions require a strong knowledge of mathematics, science or related areas (e.g., Commission of the European Communities, 2011). In educational circles, there is a concern to ensure that adequate numbers of students choose to study STEM subjects (science, technology, engineering and mathematics) at school, particularly at advanced levels (e.g., Jeffes et. al, 2012). In Ireland, a shift towards a knowledge-based economy has been signalled in government reports (e.g., Department of the Taoiseach, 2008) and policy documents (e.g., Department of the Taoiseach, 2011). These moves have been accompanied by a strong reform agenda in education, including the introduction of *Aistear*, a curriculum framework for children in preschool and in the early years of primary school (NCCA, 2009a), and revised syllabi in mathematics at post-primary level (The *Project Maths* initiative). Now the focus has shifted to mathematics at preschool and early primary levels.

Developing Mathematics Education in Ireland for Children Aged 3–8 Years

Preschool education and care in Ireland is to a large extent provided by community and voluntary agents and agencies, supported by grant aid from the government. In January 2010 a 'free preschool year' was introduced. The objective of this Early Childhood Care and Education Programme, which is open to both community and commercial service providers, is to benefit children in the key developmental period prior to starting school. Approximately 63,000 children

participated in the preschool year in the first year of its implementation. The free preschool year is now available to all eligible children in the year before they attend primary school and there is the possibility that in the near future this will be extended to two years.

Children in Ireland can be enrolled in primary schools from the age of four, and up to recently half of all four-year-olds and almost all five-year-olds were enrolled in infant classes in primary schools. Also, there are approximately 1,600 three-year-old children, deemed to be at risk of educational disadvantage, enrolled in half-day preschool sessions in Early Start units in primary schools. The Delivering Equality of Opportunity in Schools (DEIS) programme[1] extends additional supports for schools in areas of economic and social disadvantage. The DES also provides various targeted supports for young children with special educational needs.

Aistear: the Early Childhood Curriculum Framework (National Council for Curriculum and Assessment (NCCA), 2009a) provides guidance and support for all adults working with the youngest children (birth to six). Sample learning opportunities related to the themes of *Communicating* and *Exploring and Thinking* illustrate in a general way how educators can support the development of various aspects of mathematical thinking and learning with toddlers and young children. However, because *Aistear* is a framework and not a curriculum, it does not provide specific guidance related to mathematics learning and teaching. The PSMC provides guidance for teachers of children from the age of 4 years. While children attending preschools may engage in many activities which promote mathematical learning and development, there is no systematic specification of these. Preschools may choose to structure their work within a particular curriculum such as *High Scope* or *Montessori*, they may use a variation on these, or they may develop their own curriculum.

Opportunities now exist for a systematic approach to rethinking mathematics education for all children aged 3–8 years. A revised approach should address the mathematical learning of children in preschool education, and also the dual, overlapping approaches described above in relation to official guidance across the age-range. It should be based on the understanding that mathematics learning begins early in the home and needs to be supported in a structured way right from the beginning of preschool education. It should also be predicated on findings that high quality early childhood education is a critical factor in ensuring that the mathematics potential of all children is realised and that existing equity gaps are closed (e.g., Bishop & Forganz, 2007; Ginsburg, Lee & Boyd, 2008; Perry & Dockett, 2008).

1 DEIS (Delivering Equality of Opportunity in Schools) is an action plan put in place by the (now) Department of Education and Skills in 2005 to address the effects of educational disadvantage in schools. The School Support Programme (SSP) under DEIS comprises a set of measures that provides schools with additional human and material resources to tackle educational disadvantage, in schools with the highest levels of assessed disadvantage. Urban schools in the SSP are allocated to Band 1 or Band 2, depending on their level of disadvantage. There is a separate set of measures for rural schools.

Curriculum Context

The current Primary School Curriculum (PSC) (Government of Ireland, 1999) was introduced in 1999, with in-service for mathematics provided in 2001–02, and implementation beginning in 2002–03 (DES, 2005). While maintaining some important links with its predecessor, *Curaclam na Bunscoile* (DE, 1971), the PSMC also drew heavily on Vygotskian ideas about teaching and learning, in that it emphasised the social aspects of mathematics development, the importance of language in acquiring mathematical knowledge, and the key role of the teacher in modelling and supporting children's emerging understanding of mathematics.

The PSMC, which is based on socio-constructivist and guided-discovery theories of learning, aimed to equip children with a positive attitude towards mathematics, to develop problem-solving abilities and the ability to apply mathematics to everyday life, to enable children to use mathematical language effectively and accurately, and to enable them to acquire an understanding of mathematical concepts and processes, as well as proficiency in fundamental skills and basic number facts.

The PSMC was generally well-received by teachers. In a review of curriculum implementation by the NCCA (2005), a majority of teachers reported an increased emphasis on practical work as its greatest success, while enjoyment of mathematics by children was also highlighted. The implementation of practical activities on a daily basis, especially for Measures[2], was also noted. About half of teachers reported that catering for the range of children's mathematical abilities represented their greatest challenge, with inadequate instructional time contributing to this. Significantly, teachers of junior and senior infants identified classifying, matching and ordering as the strand units that had most impact on their planning and teaching[3]. Data was identified as the strand that teachers struggled with most often.

An evaluation of curriculum implementation by the Inspectorate of the then Department of Education and Science (DES, 2005), which was mainly based on observations of the work of teachers in teaching mathematics in school settings, found that the PSMC was not being implemented successfully in a significant minority of schools and classrooms. For example, some difficulties with implementation were noted, especially for Shape and Space, where children in one-third of observed classes were able to name shapes, but were not familiar with their properties, and for Data, where scope was identified for the development of specific skills and the use of integration, linkage, and a stronger cross-curricular approach. In the case of teaching

2 Measures is one of five strands in the curriculum for all grade bands. The others were Number, Shape & Space, Algebra and Data.

3 Whereas the PSMC has five strands at all grade bands, an additional strand – Early Mathematical Activities – is included in the curriculum for junior and senior infant classes, and its strand units are Classifying, Matching, Comparing and Ordering.

Problem-Solving[4], where weaknesses were also apparent in one-third of classrooms, inspectors referred to non-implementation of the school plan with respect to problem-solving, and 'an over-reliance on traditional textbook problems, which did not promote the development of specific problem-solving skills' (p. 29). In considering the use of guided discovery methods and concrete materials, inspectors noted that learning in one-third of classes 'was passive and reliant on activities that lacked focus and required more purposeful direction by the teacher' (p. 29). Inspectors also noted an 'over-reliance on whole-class teaching, where teacher talk dominated and where pupils worked silently on individual tasks for excessive periods' (p. 30). A number of difficulties with the assessment of mathematics were also noted, including inappropriate use of standardised tests and an absence in some classrooms of continuous records of children's achievement.

Performance Context

A number of studies, both national and international, have raised concerns about performance among children in Ireland on specific aspects of mathematics, and, in some cases, on overall mathematical performance.

The 2009 National Assessment of Mathematics Achievement (Eivers et al., 2010) – the first national assessment since the introduction of the PSMC to assess mathematics in both second and sixth classes – reports poorer performance on items designed to assess Measures at both class levels, compared with other content strands, and a decline in performance on Measures and Shape and Space between second and sixth; performance on items designed to assess the Applying and Problem-Solving process skill was weak at both second and sixth classes (a finding that also emerged in earlier national assessments conducted at other grade levels). Other problematic areas were integrating mathematics into other subject areas (61%), working with lower-achieving children in mathematics (60%), and extending the abilities of higher-achieving children (56%).

In the mathematics component of the Trends in International Mathematics and Science Study (TIMSS), administered to children in fourth class in over 50 countries in 2011, Ireland achieved a mean score (527) that was significantly above the international average, but significantly below the mean scores of 13 countries/regions, including Northern Ireland (562), Belgium (Fl.) (549), Finland (545), England (542), the United States (541), the Netherlands (540) and Denmark (537), as well as several Asian countries. Further, whereas 9% of children in Ireland achieved at the Advanced International Benchmark (the highest 'proficiency' level on TIMSS maths), 43% of children in Singapore, 39% in Korea and 24% in Northern Ireland did so (Eivers & Clerkin, 2012). Relative to their performance on the test as a whole, Irish children performed quite well on the TIMSS content area of Number, and less well on Geometric Shapes & Measures

4 Applying and Problem-Solving is one of six process skills in the PSMC which are taught at all grade bands. The others are Communicating and Expressing, Integrating and Connecting, Reasoning, Implementing and Understanding and Recalling.

and on Data. On the process subscales, children in Ireland performed relatively well on Knowing items, and quite poorly on Reasoning items, including items requiring problem-solving abilities.

The relatively disappointing performance of children in Ireland on TIMSS mathematics contrasts with the performance of the same children on a related study administered at the same time – the 2011 Progress in International Reading Literacy Study (PIRLS). Just five countries had mean scores that were significantly higher than Ireland's in PIRLS and the proportion of high achievers in Ireland was about the same as in other high-scoring countries (Eivers & Clerkin, 2012).

In 2009, 15-year-olds in Ireland performed significantly below the average for OECD countries on the mathematics component of the Programme for International Student Assessment (PISA), ranking 26th of 34 OECD member countries. Further, 21% of students in Ireland performed at or below Level 1 on the PISA mathematics scale. This is interpreted by the OECD (2010) as indicating that they lack the mathematics skills needed for everyday living and/or future study. While the size of the decline in performance on PISA mathematics in Ireland between 2003 (503 points) and 2009 (487) has been disputed (Perkins et al., 2012), the relatively disappointing performance by children in Ireland in mathematics across PISA cycles is worth noting, in a context in which performance on reading literacy (except in 2009) and scientific literacy have been above their respective OECD averages. Concern must also be expressed at the relatively poor performance of students in Ireland on the Space and Shape component of PISA mathematics in 2003 and 2012, when their mean scores were significantly below the corresponding OECD average. PISA Shape and Space items require students to solve problems that include shapes in different representations and dimensions (Cosgrove et al., 2005; Perkins et al., 2013). Female students in Ireland performed particularly poorly on this PISA content domain.

Notwithstanding differences between national curricula/syllabi and the assessment frameworks accompanying international studies (e.g., Close, 2006), the relatively disappointing overall performance of children in Ireland on international studies of mathematics achievement is a matter of concern, given current concerns about standards in mathematics, the role of mathematics in other subjects, and efforts to encourage students to select STEM subjects, especially at upper-secondary level. Related to this, it is a matter of concern that problem-solving presents a significant difficulty for children in Ireland from at least second class onwards. Without a strong foundation in this important process, many children may not reach their potential in mathematics.

Policy Context

There have been two significant policy initiatives in mathematics education in recent years. The first, *Project Maths* (PM), a new initiative to change the teaching and assessment of mathematics in post-primary schools, has been underway on a phased basis since 2008, and many aspects of the revised PM syllabi have now been implemented in all schools. The broad aims of PM, which is based on sociocultural theories of mathematics, are to equip students at Junior and Leaving Certificate levels with:

- the ability to recall relevant mathematical facts
- instrumental understanding ('knowing how')
- relational understanding ('knowing why')
- the ability to apply their mathematical knowledge and skill to solve problems in familiar and unfamiliar contexts
- analytical and creative powers in mathematics
- an appreciation of mathematics and its uses
- a positive disposition towards mathematics (Government of Ireland, 2012, p. 6).

Important features of *Project Maths* include the following:

- an acknowledgement of the continuum of mathematics development that extends from early childhood through post-primary schooling, with an emphasis on connected and integrated mathematical understanding
- efforts to establish links between mathematics learning at primary level through the implementation of a common introductory course in the first year of post-primary schooling
- establishment of a learning environment for problem-solving, in which problem-solving permeates all aspects of mathematics learning, and students consolidate previous learning, extend their knowledge, and engage in new learning experiences
- engagement with a wide variety of mathematical problems, some of which are purely mathematical, and others more applied
- links within strands of study to other subjects
- a focus on conceptual understanding (Government of Ireland, 2012, p. 8).

The effects of PM are as yet unclear. An initial study (Jeffes et al., 2012) was somewhat positive about the performance of samples of Junior Certificate and Leaving Certificate students[5] on tests of mathematics administered in spring 2012 that were benchmarked against international standards. However, no significant differences in performance were found between students in initial PM schools (where implementation of PM began in 24 schools in 2008) and other schools (where implementation began in 2010). Nevertheless, students in the initial schools were found to engage more often in the types of activities associated with PM (applying mathematics to real-life

5 Junior Certificate students in other schools had not studied any of the new syllabus materials at the time of testing; Leaving Certificate students in other schools had studied some aspects.

situations, conducting investigations, and participating in discursive and collaborative activities), compared with students in schools where PM had not been fully implemented. A follow-up report on the implementation of PM (Jeffes et al., 2013) again raised issues about the extent to which PM was being implemented effectively in schools. A review of student materials found evidence of a strong emphasis on implementing mathematical procedures and, to a lesser extent, problem solving, but 'little evidence that students are demonstrating reasoning and proof and communication, or making connections between mathematics topics' (p. 5).

The relevance of PM to mathematics in early years and primary school settings concerns the extent to which proposals for change among children in the 3–8 years range might need to be broadly consistent with the goals and methodologies underpinning *Project Maths*, including a substantially-increased emphasis on problem-solving, and a strong focus on the application of mathematical ideas in real-life contexts. In considering this it can be noted that these two elements are key features of the current PSMC.

In the second policy initiative, the Department of Education and Skills published a *National Strategy to Improve Literacy and Numeracy Among Young People 2011–20* (DES, 2011). The strategy made a strong case for improving standards in literacy and numeracy across all levels of the education system, and set out a series of actions designed to bring about improvement, including

- an increased focus on literacy and numeracy across the curriculum, including increased allocation of time to the teaching of English and mathematics, some of which could involve cross-curricular activities
- the clear specification of learning outcomes in revised curricula and the provision of exemplars to illustrate such outcomes
- the extension of the *Aistear* early childhood framework (NCCA, 2009a; 2009b) to children in the 4–6 years age range (i.e., those in the infant classes in primary school)
- the achievement of specified targets in the *National Assessment of Mathematics* at second and sixth class (an increase in the proportion of children achieving at the highest proficiency levels, and a reduction in the proportion achieving at the lowest levels)
- the achievement of an increase in the percentage of students taking the Higher Level mathematics examination in Leaving Certificate to 30% by 2020 (it was 22% in 2012).

These actions, together with a range of related measures in the areas of teacher education and teacher professional development, are intended to result in a significantly changed educational landscape over the next few years, compared with that in place when the PSC was introduced in 1999.

Linguistic and Social Contexts

Significant demographic changes have occurred in Ireland since the PSC was introduced in 1999, including greatly increased participation of children in the education system who do not speak the language of instruction (English or Gaeilge) at home. In the 2009 *National Assessment of Mathematics*, 15% of children in second class who were born outside Ireland had a mean score that was lower than that of Irish-born children, but the difference was not statistically significant (Eivers et al., 2010). However, 10% of children (mainly born outside of Ireland) reported speaking a language other than English or Irish most often at home, and these children had a significantly lower mean score (by 22 points) than speakers of English. Interestingly, the difference between the latter groups at sixth class was 12 points, and it was not statistically significant. These outcomes point to challenges faced by children who speak a language other than English at home in learning mathematics. They also point to the need to develop language in the context of teaching mathematics, and suggest that progress can be made over time.

Another linguistic context is that in which children learn mathematics through the medium of Irish. In a study of mathematics performance in Irish medium schools in 2010, children in second class in Gaelscoileanna achieved a mean score that was significantly higher, by one-sixth of a standard deviation, than the average score obtained by a national sample in the 2009 National Assessment (Gilleece Shiel, Clerkin, & Millar, 2012). However, by sixth class, children in Gaelscoileanna had a mean score that was not significantly different from the national average.[6] The latter result is particularly interesting as the same children achieved a mean score in English reading that was one-third of a standard deviation above the national average – with the strong performance in English reading attributed to higher socio-economic status among children in Gaelscoileanna.

Gilleece et al. also found that children in Gaeltacht schools in which Irish was the medium of instruction achieved a mean score that was not significantly different from the national average in second class, and was significantly higher in sixth class.

The outcomes of this study point to the challenges faced in teaching mathematics in Irish-medium contexts, and to issues around assessment of mathematics, including the language of assessment (i.e., whether children are assessed in Irish, English, or a combination of the two languages) and instruction (whether, to what extent, and how English is used in mathematics classes).

Socio-economic status has been identified as a factor associated with mathematics achievement. In the 2009 National Assessment of Mathematics Achievement, children in second class attending DEIS Band 1 urban schools (those with the highest levels of socio-economic disadvantage) achieved a

6 In second class, 91% of pupils in Gaelscoileanna and 49% in Gaeltacht schools completed the test in Irish. The corresponding figures for sixth class were 81% and 59% respectively. It is unclear whether all pupils in sixth class taking the test through Irish were able to demonstrate the full range of their mathematical competencies.

mean score (217) that was lower than the mean score of children in DEIS Band 2 schools (228), and significantly lower than children in non-DEIS urban schools (253) (Eivers et al., 2010). Children attending DEIS rural schools (266) and children attending non-DEIS rural schools (259) also achieved scores that were significantly higher than children attending DEIS Band 1 schools. Outcomes were broadly similar at sixth class, where there was also a difference of 40 points (four-fifths of a standard deviation) between children in DEIS Band 1 urban schools and those in non-DEIS urban schools.

There is evidence that some of the differences in mathematical achievement found in school settings may have their origins in children's home backgrounds. In TIMSS 2011, fourth class children in Ireland and on average across participating countries who had 'some' or 'few' human resources at home[7] achieved a mean mathematics score that was significantly lower (by one-half of a standard deviation) than that of children with 'many' resources (Mullis, Martin, Foy, & Arora, 2012). The relationship between home environment and mathematics achievement may be mediated by the types of language and mathematical activities – whether formal or informal – in which children engage in their home. International research (e.g., Sylva, Melhuish, Sammons, Siraj-Blatchford & Taggart, 2004) indicates that it is what parents do with children at home, rather than who they are, that is of most significance to children's early learning.

International research has identified gaps in children's mathematical knowledge well before they start school, in particular among children living in disadvantaged circumstances (e.g., Jordan & Levine, 2009), with more marked differences on tasks requiring language (Jordan et al., 2006). This issue is discussed further in *Chapter 3*.

Overview of Chapters

Educators' beliefs are strongly associated with how they see mathematics and mathematics education. Thus, the opening chapter of this report presents three conceptions of mathematics and their different implications for mathematics education. It emphasises current views of mathematics education as a cultural phenomenon, where issues of equity and access are paramount and where numeracy is seen as one aspect of mathematics. The concept of mathematical proficiency is presented as an overarching aim, with mathematization as integral to its achievement.

While a wide range of theories are available for explaining mathematical learning and development during early childhood, in *Chapter 2* we focus on the perspectives afforded by cognitive and sociocultural theories. These perspectives are the ones that underpin key developments in mathematics education over the last decade. Constructionism is also highlighted because of its importance in underpinning recent developments in digital learning and in the use of digital tools for learning.

7 Level of human resources at home was based on an index that included number of children's books in the home, at least two home study supports (internet connection, own room), parental occupations and parental education.

In *Chapter 3*, the role of language and communication in young children's mathematical development is considered. Ideas about developing children's mathematical vocabulary and their engagement in math talk are elaborated on. The mathematical language needs of children in disadvantaged circumstances and those with language impairment are also considered.

Two concepts arise as we explore the task of identifying goals for early childhood mathematics education in *Chapter 4*: the concept of 'big ideas'; and that of 'powerful mathematical ideas'. Differences in emphases between the two approaches to the specification of curriculum goals are discussed. These are compared with the approach used to specify content and skills in the 1999 PSMC. The level of detail that might be employed in the specification of goals is also an issue addressed in this chapter.

Chapter 5 traces ideas about stages of development from those associated with Piaget to current conceptions of learning trajectories or learning paths. The literature shows how a cognitive perspective may give rise to interpretations of children's thinking about mathematical concepts as predetermined. This is contrasted with a sociocultural/situative stance where changes in levels of understanding are explored in order to clarify the particular paths that children take. We discuss the implications of these different perspectives for learning and teaching.

The range of methods for the formative assessment of children's mathematical learning is reviewed in *Chapter 6*, including observations, tasks, interviews and conversations. Consideration is also given to the appropriateness of using more formal assessments, including screening/diagnostic assessments. Potential difficulties relating to the use of standardised tests with children in the age range of 3–8 years are highlighted. This chapter also considers assessment of children with special educational needs and those for whom their first language is not the language used in the education setting.

Chapter 7 focuses on how preschools and schools might address equity issues in learning mathematics. The perspective we present is that groups of children identified as at-risk of underachieving because of a learning disability or talent do not require distinctive teaching approaches, but that account needs to be taken of their individual learning needs. We also identify other groups who may appear to be underachieving because of cultural/social factors, and suggest how provision can be made to address their particular needs.

Chapter 8 outlines the key implications for the redevelopment of the PSMC for children from 3–8 years of age arising from the report.

CHAPTER 1

Defining Mathematics Education

Mathematics learning begins from birth as children explore the world around them. As they develop, they are supported in their learning by the people around them. The environment is a rich resource for engaging with mathematics, especially when it provides opportunities to listen to and use mathematical language and to engage in mathematical ways with everyday experiences. Through the assistance of others, children's attention and activity are directed in ways that enable them to reason and to grow in their abilities to communicate mathematically. As they do so, they develop an affinity with mathematical tools and they take pleasure and interest in thinking mathematically.

In order to facilitate children's journeys into the world of mathematics and to afford them a rich experience of the subject, it is important to give consideration to issues related to the foundations of mathematics, what it means to engage in mathematics and the key aims and goals for mathematics education at preschool and primary levels. These matters are explored further in this chapter.

The Foundations of Mathematics

Davis and Hersh (1981) suggest that three standard dogmas are usually presented in discussions on the foundations of mathematics – Platonism, formalism and constructivism. Platonists are of the view that mathematical objects (e.g., geometric shapes) are real and objective and that their existence is independent of an individual's knowledge of them. Those who adopt the formalist perspective believe that there are no mathematical objects and that mathematics comprises definitions, theorems and axioms. What matters to them are the rules and how one formula can be transformed into another. According to constructivists, mathematics is comprised only of those objects that individuals construct themselves. Those who hold this conception of mathematical knowledge view it as 'tentative, intuitive, subjective and dynamic' (Nyaumwe, 2004, p. 21). Hersh (1997) argues that each of these three views is limited, e.g., Platonism denies the human dimension of mathematics while constructivism fails to explain the universality of mathematical knowledge (see also Stemhagen, 2009) and, therefore, he adopts a humanist stance, that is:

- Mathematics is human – it is part of and fits into human culture.
- Mathematical knowledge is not infallible. Like science, mathematics can be advanced by making mistakes, correcting and re-correcting them.

- There are different versions of proof and of rigour depending on time, place and other things. For example, the use of computers in proof is a recent phenomenon.
- Mathematical objects are a distinct variety of social-historic objects. Like literature or religion, they are a special part of culture. (ibid., p. 22)

From this perspective, mathematical entities derive from the needs of everyday life (e.g., in science or technology) and have no meaning beyond that ascribed to them by a shared human consciousness.

Since the 1980s, there has been considerable interest in the relationship between teachers' conceptions of mathematics and their pedagogical practices (e.g., Dossey, 1992; Ernest, 1989; Leder, Pehkonen, & Törner, 2003). Ernest (1989) argues that a teacher's conceptions or set of beliefs about the nature of mathematics as a whole forms the basis of his/her philosophy of mathematics. However, these conceptions are not necessarily consciously held views or fully articulated philosophies. Drawing on the work of Thompson (1984), Ernest (1989) identified three conceptions of mathematics held by teachers:

1. Platonist: view of mathematics as a static but unified knowledge that can be discovered rather than created.
2. Problem-solving: view of mathematics as on-going enquiry and coming to know.
3. Instrumentalist: view of mathematics as a 'bag of tools' made up of utilitarian facts, rules and skills.

These conceptions of mathematics align with philosophies of mathematics as described above (Dossey, 1992; Ernest, 1989). They have different implications for mathematics education. Most notably, the view of mathematics as 'absolute and certain' is often perceived as eliminating learners, particularly women and marginalised groups, from the subject – '[n]ot only is the personness of the discipline removed, but hierarchy of knowledge and elitism of knowers construes an antagonistic cultural climate in classrooms' (Burton, 2001, p. 596). On the other hand, a view of mathematics as co-constructed promotes student engagement and critical thinking (e.g., Povey, 2002; Stemhagen, 2009). Since a teacher's beliefs about mathematical content, the nature of mathematics and its teaching and learning are strongly associated with what he or she does in the classroom (e.g., Törner, Rolka, Rösken, & Sriraman, 2010), any proposed change in the curriculum rests on addressing these beliefs.

A Definition of Mathematics Education

According to Valero (2009), mathematics education can refer to two different domains: a field of practice where people engage in the activities connected to the teaching and learning of mathematics and a field of study which is the space of scientific enquiry on and theorisation about the field of practice. It is his contention that the field of study defines the field of practice and since the former is often focused on the relationship between teacher, learner and mathematical content,

broader social, cultural and political factors are overlooked. He argues for broadening the scope of mathematics education:

> *Let us think about mathematics education as a field of practice covering the network of social practices carried out by different social actors and institutions located in different spheres and levels, which constitute and shape the way mathematics is taught and learned in society, schools and classrooms…This broader definition of the field evidences the social, political, cultural and economic dimensions that are a constitutive element of mathematics education practices.* (p. 240)

Current views of mathematics education are inextricably linked with ideas about equity and access (Bishop & Forganz, 2007) and reflect this broader scope. In attempting to define mathematics education, one is forced to consider questions such as 'What mathematics?' and 'Mathematics for whom?' and 'Mathematics for what purpose?'. Bishop (1988, as cited in Bishop & Forganz, 2007), in seeking to answer the first of these questions, differentiates between Mathematics with an upper-case M and mathematics with a lower-case m, both of which, in his view, should be addressed by schools. He regards the former as the universal mathematics that is the basis of mathematics curricula in schools, while the latter refer to a wider mathematical knowledge that is used in everyday life in a particular society or culture. Current views of mathematics education assume that we are talking about mathematics education for all children. In clarifying the meaning of 'mathematics for all', Clements, Keitel, Bishop, Kilpatrick & Leong (2013) suggest that it is 'the kind of goal that anticipates a world in which all people have the opportunity to learn, and benefit from learning, mathematics' (p. 8). In response to the third question, the purposes of learning mathematics in schools are now seen as twofold: the preparation of mathematically-functioning citizens of a society and the preparation of some for future careers in which mathematics is fundamental. Bishop and Forganz (2007) state that, from an equity perspective, no one should be denied access to participation along this path. Clements et al. (2013) draw attention to the fact that, in learning mathematics, conditions and context are crucial. They, in common with others (e.g., van den Heuvel-Panhuizen, 2003), reiterate the fact that mathematics is a cultural phenomenon and that the forms of mathematics in schools should 'arise out of, and are obviously related to, the needs of learners, and the societies in which they live' (Sriraman & English, 2010, p. 33).

Numeracy

Numeracy is a term having a range of different definitions, many of which encompass the equity and access ideals of 'mathematics for all' and ideas related to competent citizenship (Bishop & Forganz, 2007). Clements et al. (2013) suggest that the concept of numeracy, while still ill-defined, has gradually been extended over the years 'beyond purely arithmetical skills to embrace not only other elementary mathematical skills but also affective characteristics such as attitudes and confidence' (p. 32).

Bishop and Forganz (2007) suggest a number of possible relationships between mathematics and numeracy:

1. Mathematics and numeracy intersect, that is they share aspects but do not include each other.
2. Numeracy is a subset of mathematics.
3. Mathematics is a subset of numeracy.
4. Numeracy is mathematics.
5. Mathematics and numeracy are two very different phenomena, having no relationship.

The term numeracy has been used in recent years by a number of governments, including those of Canada, Australia and Ireland, to describe aspirations for aspects of mathematics learning including quantitative literacy. Numeracy, as it is generally envisaged in such statements, is seen as something which is not limited to the ability to use numbers, but, for instance, as 'the capacity, confidence and disposition to use mathematics to meet the demands of learning, school, home, work, community and civic life' (DES, 2011, p. 9). This is very similar to that used by the Australian Association of Mathematics Teachers (AAMT, 1998) which states that 'To be numerate is to use mathematics effectively to meet the general demands of life at home, in paid work, and for participation in community and civic life' (p. 2).

The statement from AAMT goes on to describe the place of numeracy in the curriculum:

> *In school education, numeracy is a fundamental component of learning, discourse and critique across all areas of the curriculum. It involves the disposition to use, in context, a combination of:*
> - *underpinning mathematical concepts and skills from across the discipline (numerical, spatial, graphical, statistical and algebraic);*
> - *mathematical thinking and strategies;*
> - *general thinking skills; and*
> - *grounded appreciation of context.* (p. 2)

This suggests a view of numeracy as involving the use of mathematics, but not as the same as mathematics. The DES definition above is in the same vein. It seems then that both the DES definition and that from AAMT correspond with a view of numeracy as a subset of mathematics, Option 2 in the Bishop and Forganz list above. We conclude then that the definitions offered above are not talking about mathematics per se but rather about a subset of mathematics which is developed in school education. While the development of numeracy is important, education at all levels should encompass a broader view of mathematics.

Defining Mathematics Education for Children Aged 3–8 Years

Aistear defines numeracy as 'developing an understanding of numbers and mathematical concepts' (2009a, p. 56). It views mathematical literacy, whereby children learn to communicate using the mathematics sign system, as part of being literate. Perry and Dockett (2008) argue that numeracy, mathematical literacy and mathematics go hand-in-hand in early education settings because young children's learning takes place in the context of holistic learning experiences and in contexts that are part of their day-to-day lives:

> *The contextual learning and integrated curriculum apparent in many early childhood – particularly prior-to-school settings – ensures that there is little distinction to be drawn between numeracy, mathematical literacy and aspects of mathematical connections with the children's real worlds.* (p. 83)

Concepts of number and operations with numbers are identified as being at the heart of mathematics for young children (NRC, 2001). But prior to children developing concepts about number, mathematical thinking begins for all children with comparisons of quantity and the development of an understanding of quantity (e.g., Griffin, 2005; Sophian, 2008). This does not mean though, that curricula for early childhood should be limited to the topic of number. Rather, as children are gradually introduced to mathematics in early education settings, it should address the range of mathematical ideas that all children need to engage with in order to reach their potential in their mathematics learning. It should also encompass all of the topics of shape and space/geometry and measure, data, and algebra (e.g., Saracho & Spodek, 2008; Ginsburg, 2009a; Clements & Sarama, 2004).

There are now a number of sources that educators can look to for advice on what principles should guide mathematics education for young children. These include statements from The National Association of Educators of Young Children (NAEYC) in the United States who joined forces with the National Council of Teachers of Mathematics (NCTM) to issue a position paper (2002/2010) on early childhood mathematics. Similarly, in Australia Early Childhood Australia and the Australian Association of Mathematics Teachers set out their position on what mathematics education for young children should be (AAMT/ECA, 2006). General principles which should underpin pedagogy/practice are explored in Report No. 18, Chapter 1, Sections: *Principles that Emphasise People, Relationships and the Learning Environment* and *Principles that Emphasise Learning*.

A Key Aim of Mathematics Education: Mathematical Proficiency

Mathematical proficiency has been adopted as a key aim in policy documents on mathematics in many countries, for example, the US (CCSSM/NGA, 2010), New Zealand (Anthony & Walshaw, 2007) and Australia (National Curriculum Board, 2009). Mathematical proficiency comprises the following five interwoven strands (NRC, 2001, pp. 116–133):

- conceptual understanding – comprehension of mathematical concepts, operations, and relations

Individuals who have a conceptual understanding of mathematics know more than isolated facts. They have an integrated grasp of mathematical ideas and know why and in what context the ideas are applicable. They make connections between ideas, thus allowing them to retain facts and procedures.

- procedural fluency – skill in carrying out procedures flexibly, accurately, efficiently, and appropriately

Individuals who are procedurally fluent in the domain of number are able to analyse similarities and differences between methods of calculating. These methods include written procedures, mental methods and methods that use concrete materials and technological tools.

- strategic competence – ability to formulate, represent, and solve mathematical problems

Individuals who are strategically competent have the capacity to form mental representations of both routine and non-routine problems, and detect mathematical relationships, and are flexible in their problem-solving approaches. Strategic competence depends upon and nurtures both conceptual understanding and procedural fluency.

- adaptive reasoning – capacity for logical thought, reflection, explanation, and justification

A hallmark of adaptive reasoning is the justification of one's work. This justification can be both formal and informal. Individuals clarify their reasoning by talking about concepts and procedures and giving good reasons for the strategies that they are employing. Such justification is supported by collaboration with others and by the use of physical and mental representations of problems.

- productive disposition – habitual inclination to see mathematics as sensible, useful, worthwhile, coupled with a belief in diligence and one's own efficacy.

Individuals who have a productive disposition believe that mathematics is useful and relevant. They do not regard mathematics as being for the 'elite few' but rather as a subject in which all can enjoy success if they make appropriate effort.

Key to the development of mathematical proficiency is the interdependence and interconnection among the strands, demonstrated in Figure 1.1.

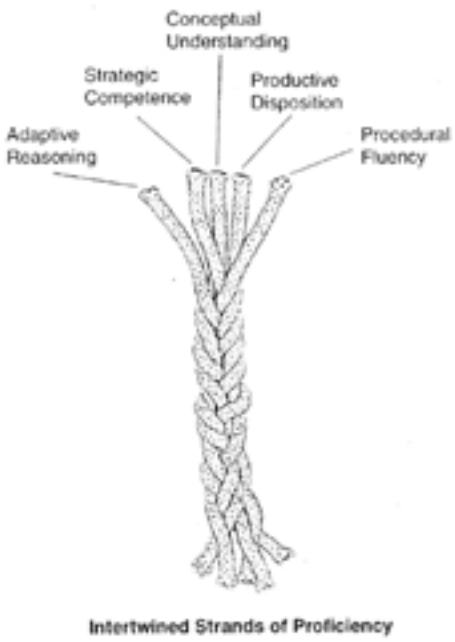


Figure 1.1: Intertwined Strands of Proficiency. From *Adding It Up: Helping Children to Learn Mathematics.* National Research Council (NRC) (2001, p. 5). Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: The National Academies Press.

Given the breadth and depth of the concept of mathematical proficiency, we support its inclusion as a key aim for the revised curriculum. As we understand it, individuals become mathematically proficient over their years in educational settings. Each of the strands becomes progressively more developed as children's mathematical experiences become increasingly sophisticated.

As described above, mathematical proficiency is developed through engagement with processes such as communicating, reasoning, argumentation, justification, generalisation, representing, problem-solving, connecting and communicating. All of these are encompassed in the overarching concept of mathematization (Bonotto, 2005; NRC, 2009). Below, we introduce some ways in which the concept of mathematization is defined in the literature.

Mathematization

The Realistic Mathematics Education (RME) movement is illustrative of how a particular perspective on mathematics suggests a particular way of conceptualising mathematics education. Freudenthal (1973) thought of mathematics not as a body of knowledge that had to be transmitted but as a form of human activity. For him, the learning of mathematics meant involving children in 'mathematization'

where, with appropriate guidance, they would have the opportunity to reinvent mathematics. Central to his learning theory was the notion of level-raising where what might be known informally at one level becomes the object of scrutiny at the next level. Treffers (1987) expands on level-raising by formulating the ideas of 'horizontal' and 'vertical' mathematization. In horizontal mathematization, the learner develops mathematical tools or symbols that can help to solve problems situated in real-life contexts. In vertical mathematization, the learner makes connections between mathematical concepts and strategies, that is, she or he moves within the world of mathematical symbols.

In the United States, the NRC report (2009) addressed the connection between mathematizing and mathematical processes:

> *Together, the general mathematical processes of reasoning, representing, problem solving, connecting, and communicating are mechanisms by which children can go back and forth between abstract mathematics and real situations in the world around them. In other words, they are a means of both making sense of abstract mathematics and for formulating real situations in mathematical terms – that is, for mathematizing the situations they encounter.* (p. 43)
>
> *Mathematizing happens when children can create a model of the situation by using mathematical objects (such as numbers or shapes, mathematical actions (such as counting or transforming shapes), and their structural relationships to solve problems about the situation. For example, children can use blocks to build a model of a castle tower, positioning the blocks to fit with a description or relationships among features of the tower, such as a front door on the first floor, a large room on the second floor, and a lookout tower on top of the roof. Mathematizing often involves representing relationships in a situation so that the relationships can be quantified.* (p. 44)

Ginsburg (2009a) argues that early childhood mathematics education should focus on mathematization. In his view, the educator's role is to support children in their efforts to mathematize. This involves 'helping them to interpret their experiences in explicitly mathematical form and understand the relations between the two' (p. 415). Often this support is offered in the course of everyday activities. The process of mathematization is also emphasised by others as a key aspect of early mathematics education (e.g., Perry & Dockett, 2008).

It follows then that mathematization fosters mathematical proficiency and so should be a key focus of early mathematics education.

Conclusion

High-quality mathematics education for children aged 3–8 years is predicated on opportunities for rich engaging interactions with knowledgeable educators who challenge children to think and communicate mathematically. They offer support for children's mathematizing, for their constructions of a good number sense and for their developing understandings of critical mathematical ideas. Educators use their knowledge of mathematics, of children's learning and of mathematics pedagogy to introduce children gradually to a structured curriculum which emphasises the development of mathematical proficiency. However, the implementation of such a curriculum is strongly linked not only to a teacher's beliefs about and attitudes towards mathematics, but also to those of the broader social arena. In particular, a view that mathematics is a human endeavour, deriving from the needs of everyday life, underpins the notion of 'mathematics for all'. Changes in the mathematics curriculum, therefore, depend on stakeholders in education engaging in conversations about mathematics education and its key aims and goals.

The key messages arising from this chapter are as follows:

- Mathematics is no longer considered to be a fixed, objective body of knowledge. Rather it comprises a number of social practices that are negotiated by the learner and teacher within the broader social, political and cultural arena. A teacher's conceptualisation of mathematics and what it is to do mathematics have strong influences on pedagogic practices.
- Current views of mathematics education are inextricably linked with ideas about equity and access. While the development of numeracy is important, a broad interpretation of mathematics should underpin efforts towards curricular reform in Ireland. A broad perspective is coherent with a view of mathematics as a human, socially-constructed and creative endeavour.
- Given the breadth and depth of the concept of mathematical proficiency, we propose that it be adopted as a key aim of the mathematics curriculum. It is promoted through engagement with processes such as connecting, communicating, reasoning, argumentation, justifying, representing, problem-solving and generalising. All of these are encompassed in the overarching concept of mathematization. Thus mathematization should be a key focus of mathematics education.

CHAPTER 2

Theoretical Perspectives

For many years psychological perspectives dominated conceptions of how children learn mathematics. A major development in the 1990s occurred with the social turn in mathematics education research (Lerman, 2000). This resulted in the increasing use of sociocultural theories to explain mathematical learning and development and a move away from seeing learning as acquisition of knowledge, to seeing learning as the understanding of practice (in this case, the practice of doing mathematics). In addition, a number of new perspectives have become visible recently, including social justice theory, networking theories and semiotics to name but a few. The emergence of all of these new ways of thinking about mathematics learning and the factors that influence it means that it is increasingly challenging to explain mathematics learning by reference to a narrow range of theories. Consequently, from the point of view of mathematics educators, a wide range of theories serve to explain children's mathematical learning and development and to influence mathematics education (e.g., Sriraman & Nardi, 2013).

Learning in early childhood, as at any age or stage of life, is generally considered to be a complex process not easily explained by a single theory or perspective (e.g., Dunphy, 2012). Within the field of early childhood education, social constructivist perspectives take account of the central role of social interaction in shaping learning. Sociocultural theories of learning, in addition to the social aspect, also consider culture and cultural influences as centrally important to learning. Cognitive perspectives arising from, for example, constructivist theories, are also useful because they emphasise the active, constructivist nature of human learning and development and the idea that we each construct our own learning.

In *International Trends in Post-Primary Mathematics Education* for the NCCA, Conway and Sloane (2005) identified three main theoretical perspectives on learning that have had a significant impact on mathematics education over the past hundred years. These included behavioural, cognitive and sociocultural theories. Behaviourist theories (which emphasise behaviour modification via stimulus response and selective reinforcement), while still influential in certain teaching practices, are no longer influential in mainstream mathematics research.

In this chapter we propose to build on Conway and Sloane's work by discussing sociocultural and cognitive perspectives as they pertain to young children's mathematical learning and development. These perspectives are central since they are the main perspectives underpinning recent significant research and developments in early childhood mathematics. The developments we refer to are discussed in both this report and Report No. 18. They include the current attention being given to curriculum goals (see *Chapter 4* in this report; Report No. 18, Chapter 3, Section: *Curriculum Goals*), learning and teaching paths in early childhood mathematics (see *Chapter 5* in this report; Report No. 18, Chapter 3, Section: *Content Areas*), as well as developments in pedagogy and assessment (see *Chapter 6* in this report; Report No. 18, Chapter 2, Section: *Meta-Practices*). These developments and their implications for the revision of the mathematics curriculum are discussed in later sections of this report. We also draw attention to the theory of constructionism, a theory of learning which takes cognisance of the role of cultural tools while also being consistent with cognitive and sociocultural theories. Constructionism's importance in this report is that it underpins the discussion in Report No. 18 of the use of ICT in the curriculum (see Report No. 18, Chapter 2, Section: *Digital Tools*).

In terms of mathematics learning and development, when the intention is to consider the progress and activity of individual learners a social constructivist/cognitive perspective is helpful, but when the intention is to focus, on, for example, teaching practices, a sociocultural perspective is appropriate. Cobb and Yackel (1996) support this pragmatic view and emphasise the use of the perspective which is most helpful for the purpose:

> *The sociocultural approach…focuses on the social and cultural bases of personal experience, whereas analyses developed from the emergent [cognitive] perspective account for the constitution of social and cultural processes by actively cognisizing individuals.* (p. 188)

By focusing on cognitive and sociocultural perspectives, we provide ourselves with different lenses with which to view mathematics learning and the pedagogy that can support it. Speaking of how theory is used to investigate and explain the complexity of human learning of mathematics, Lerman (1998) describes different perspectives as the zoom of a lens. The focus can be on mathematical tasks, representations and inscriptions, on problems and problem-solving, on the individual or the group, on the interactions between them, on communication and gesture and all the contexts in which these occur.

Sociocultural Perspectives

Sociocultural theories emphasise the social and cultural as inseparable contexts in which learning can be understood. They are sometimes referred to as cultural-historical theories, in order to explain the role that the past is seen to play in present culture and in social interactions. Sociocultural theories are increasingly the dominant framework used in early childhood education to explain

young children's learning (NCCA, 2009a). Sociocultural theories include the range of Vygotskian and post-Vygotskian theories. Vygotskian theory argues that learning is socially mediated from the beginning. Notions such as 'interactions', 'shared attention' and 'intersubjectivity' are crucial. Bodrova and Leong explain that 'A mental function exists or is distributed between two people before it is appropriated and internalised' (2007, p. 79). Shared activities and shared talk are essential contexts within which learning occurs. Key sociocultural theorists such as Rogoff and Bruner also take a sociocultural approach to learning.

Rogoff (1998, p. 691) describes learning or development as a transformation of participation. From her perspective, transformation occurs at a number of levels: for instance, the learner changes at the level of their involvement, in the role they play in the learning situation, in the ability they demonstrate in moving flexibly from one learning context to another, and in the amount of responsibility taken in the situation. Learning is seen as a process by which children change as a result of taking part in activity. They become more able, and they participate with increasing confidence in similar activities. Children change both in their understanding of the activity and in terms of their role in the activity. Rogoff emphasises the personal, interpersonal and community aspects of the learning situation. The community aspect draws attention to culture, the interpersonal aspect draws attention to the interactions that are part of the learning process and the personal aspect draws attention to transformations in individuals' participation in activity. This perspective is coherent with Bruner's views.

Bruner's (1996) sociocultural theory of learning suggest that the process of learning is as much a social construction as it is an individual one – 'human mental activity is neither solo nor conducted unassisted, even when it goes on 'inside the head' (p. xi). In his view, culture shapes minds as 'it provides us with the toolkit by which we construct not only our worlds, but our very conceptions of ourselves and our powers' (p. x). In seeking to understand learning, Bruner argues that

> *…you cannot understand mental activity unless you take into account the actual setting and its resources, the very things that give mind its shape and scope. Learning, remembering, talking and imagining: all of them are made possible by participating in a culture.* (pp. x-xi)

Agency, collaboration, reflection and culture are four crucial ideas for learning identified by Bruner. He emphasises the role of language in the functioning of the mind and school as a culture itself, not just a preparation for it. He sees interactions between the learner and more experienced others as crucial to learning. More experienced others scaffold learning. The tools, physical and cognitive, that are used by people to assist in making and sharing meaning are considered by Bruner (1966) to be highly significant in determining learning. Some tools enhance action, others enhance the senses while still others enhance thought. The expectation is that highly abstract uses of symbolic forms and language – both spoken and written – are generally developed in schools.

Learners appropriate or internalise cultural tools (e.g., language, computers, numbers) to their own activity. Internalisation means 'knowing how', while appropriation means taking a tool and making it one's own. However, in terms of learning mathematics, this doesn't mean appropriation of the ideas of others: rather it means learners gradually transform initial ideas into fully-developed mathematical concepts under the influence of interaction with adults. From the sociocultural perspective, there is a back and forth relationship between notations-in-use and mathematical sense making, 'cultural conventions such as notational systems…shape the very activities from which they emerge, at the same time that their meanings are continuously transformed as learners produce and reproduce them in activity' (Meira, 1995, p. 270). In early childhood, children initially develop their own marks and representations to communicate their mathematical thinking. These mathematical graphics can include, for example, scribbles, drawings and invented symbols and perhaps numerals and letters. Critically, these lay down the foundations for the later use of standard forms of written mathematics (Carruthers & Worthington, 2006). Perry and Dockett (2008) suggest that children develop their own symbol systems first, and use this knowledge until another, more standardised system, can be taken on board. Interactions with more knowledgeable others are particularly important since it is as a result of interactions about the meanings of marks and symbols (their own and the more conventional ones) which enable children to learn about the meaning and roles of mathematical symbols. This can be compared with encouraging young children to use their own strategies and methods to solve mathematical problems. Hence, children can be encouraged to use their own language, at least in stages where their concepts are being formed.

A Cultural-Historical Activity Theory Perspective

Activity theory is a development of aspects of Vygotsky's work (e.g., Engerström et al., 1999). Modern developments of activity theory are known as cultural-historical activity theory (CHAT) and these are characterised as a framework rather than as a theory with a set of neat propositions (Roth & Lee, 2007). Activity theory has been influential, particularly in relation to language, language learning and literacy but its implications for mathematics learning are only now being articulated. The framework focuses on culture, diversity, multiple voices, communities and identity (Ryan & Williams, 2007).
It focuses on the joint activity in the learning situation, rather than on individual learners: 'a communal activity shared by a group typically has a communal 'object'. In schooling we might say the object is the 'task' to be carried out by the children and teacher' (p. 162). Activity theorists claim that making activity the focus results in a holistic view of learning (e.g., Roth & Lee, 2007, p. 218). Children use tools such as language, a particular action or resource to mediate knowledge in interactions with others. Ryan and William see potential in the way CHAT helps us to view the relationships between everyday activity and school mathematics and the role that everyday mathematics can play as a boundary object between the two. It also has potential for offering opportunities for shared learning and for the analysis of how this affects individual learning (Roth & Lee, 2007).

A Situative Perspective

Situative cognition sees understanding as situated in the body, in space and time, as well as socially and culturally. For instance, Ryan and Williams (2007) describe how situative theorists (e.g., Lakoff & Nunez, 2000) have analysed the number line from an embodied cognition point of view and how those authors see the number line as a particularly powerful model in that 'it allows the learner to situate themselves bodily and spatially in the mathematics in a powerful way' (p. 19). For example, young children can explore number relations and operations on a floor number line, by moving themselves forward and back on the line. Studies of out-of-school learning have revealed the situated nature of mathematical practices and of mathematical learning. For example, Nunes, Carraher and Schliemann (1993) have compared Brazilian children's facility with 'street mathematics' with their achievement in 'school mathematics'. From the situative perspective, learning takes place in the same context in which it is applied. This implies that it is important to think about the context in which learning takes place, all the constraints and affordances governing the site of learning and the use the learner makes of these (Greeno, 1991; 1997). When situated cognitionists speak of context, they are referring to a social context, defined in terms of participation in social practices (Lave, 1988). The social engagements that enable learning are a key focus. A number of studies in mathematics learning have indicated that different forms of mathematical reasoning arise in the context of different practices (e.g., Cobb & Bowers, 1999). The implication of this is that, if educators wish to encourage children's argumentation and reasoning, attention must be paid to the practices that are put in place to support these processes (see Report No. 18, Chapter 3, Section: *Mathematical Processes*).

Two theorists have worked separately (Lave, 1988; Wenger, 1998) and together (Lave & Wenger, 1991) to conceptualise a theory of learning which has given rise to notions of learning by belonging. They introduce the notion of legitimate peripheral participation as a pathway to learning in a community of practice. The practices of the community constitute what is to be known, learning is about participating more fully in the practices and moving from the periphery to the centre of practice (becoming more able). The idea that 'developing an identity as a member of a community and becoming knowledgeably skilful are part of the same process, with the former motivating, shaping and giving meanings to the latter, which is subsumed' (Lave, 1988, p. 65) can be used as a way of thinking of classrooms as mathematics learning communities.

To summarise, there are a number of implications for early mathematics education arising from sociocultural theories. For instance, interaction and collaboration with others is central. Culture plays a key role in learning; both the culture the children bring to the setting and the culture of the setting. This provides the context for learning. Children's agency is recognised, as is their strong interest in dialogue and discourse with others. Collaborating and establishing joint understanding are important. Establishing a zone of proximal development, within which to guide and support learning is a key task for the proactive educator. As well as scaffolding learning, the educator engages in the co-construction of meaning with the child, based on awareness and understanding

of the child's perspective. Preschools and classrooms are seen as communities of practice where children learn mathematics as they engage with their teachers and peers in everyday activity in these settings.

Cognitive Perspectives

Cognitive theorists focus on internal cognitive structures and view learning as changes in these structures. Cobb (2007) emphasises cognitive psychologists' interest in how change occurs, most significantly qualitative changes in learners' mathematical reasoning. He identifies two general types of theories within the cognitive science tradition that relate to specific domains: theories which offer insights into the processes of children's learning and theories of the development of children's reasoning.

Constructivist Perspectives

Most of the current theorising about mathematical learning and development is grounded in Piaget's constructivism, a theory which emphasised the active construction of knowledge by learners through processes of assimilation and accommodation, in interaction with the environment. During the 1970s and 1980s, the Piagetian influence on mathematics education was enormous (Anderson, Anderson, & Thauberger, 2008). Through these decades various forms of constructivism were developed and there were ensuing conflicts, challenges and efforts at synthesis. Fosnot (1996), drawing from the work of various theorists, defines constructivism as

> *a theory about knowledge and learning; it describes both 'knowing' and how one 'comes to know'. Based on work in psychology, philosophy and anthropology, the theory describes knowledge as temporary, developmental, non-objective, internally constructed, and socially and culturally mediated. Learning from this perspective is viewed as a self-regulatory process of struggling with the conflict between existing personal models of the world and discrepant new insights, constructing new representations and models of reality as a human meaning-making venture with culturally developed tools and symbols, and further negotiating such meaning through cooperative social activity, discourse and debate.* (p. ix)

The use of the metaphor of learning as a process of construction has been traced from Vico's 18th century philosophical writings, to those of Kant in the 19th century. More recently, theorists such as Von Glasersfeld (1984) and Steffe (1992) were seen as radical constructivists due to their more radical views of learning when compared with those of Piaget. From the perspective of radical constructivists, learning is seen as self-regulation and self-organisation (e.g., Hufferd-Ackles, Fuson & Sherin, 2004). Since radical constructivism rejects the notion of an external, independent, objective reality, one aspect of individual learners' organisation is the world they construct through their experience, i.e. individuals construct their own ways of knowing (Von Glasersfeld, 1989).

Another important form of constructivism is the social constructivism of Ernest (1991) and colleagues. This is based on three grounds: linguistic knowledge, conventions, and rules form the basis for mathematical knowledge; interpersonal social processes are needed to turn an individual's subjective knowledge into accepted objective knowledge; objective knowledge is understood to be social (Sriraman & Haverhals, 2010). Socioconstructivists see as complementary the social and cognitive aspects of knowledge construction, explaining learning by drawing from both perspectives. Differences in the various forms of constructivism essentially revolve around the interplay between subjective and objective knowledge (Sriraman & Haverhals, 2010).

The psychological constructivist view of how children learn mathematics is, according to Battista (2004):

> *determined by the elements and organisation of the relevant mental structures that the students are currently using to process their mathematical worlds…To construct new knowledge and make sense of novel situations, students build on and revise their current mental structures through the processes of action, reflection and abstraction.* (p. 186)

This conception of learning mathematics is the one which underpins the learning trajectories literature which is reviewed in *Chapter 5* in this report.

Various attempts have been made to derive teaching approaches coherent with constructivist perspectives. For instance, Jaworski (1992) proposed three elements inherent in constructivist mathematics teaching: the provision of a supportive learning environment; offering appropriate mathematical challenge; and nurturing processes and strategies that foster learning. Constructivist teaching techniques are sometimes associated with 'discovery methods' and often contrasted with the explicit presentation of information to learners (e.g., Sweller, 2009). One critique of constructivist approaches is that they offer minimal guidance to learners (e.g., Kirschner, Sweller, & Clark, 2006), but this is disputed by proponents of such approaches. Duffy (2009) argues that in fact the difference in constructivist and explicit instruction approaches resides not in how or indeed how much guidance they offer to learners, but in their conception of the stimulus for learning. He considers that this is not addressed in explicit instruction approaches but in contrast is seen as central in constructivist approaches. That stimulus for constructivists is the need for learners to understand, to make sense of what it is they encounter.

Constructionism is a theory of learning which takes cognisance of the role of cultural tools, while also building on constructivism and sociocultural theories. Below we explore how this perspective underpins recent developments in digital learning and in the use of digital tools for learning.

Constructionism

The core concern of sociocultural theories is the mediated nature of all human activity through interactions with others around tasks and activities and with material and symbolic tools. From this

perspective, 'tools' are conceived in a broad sense, including not only physical artefacts but also symbolic resources such as those of natural language and technical procedures such as mathematical algorithms. Cultural tools, whether physical or symbolic, are considered to influence the ways in which people interact with and think about the world. Bruner (1973, p. 22) saw thinking as the 'internalisation of 'tools' provided by a given culture' while Vygotsky (1978) saw changes in tools as bringing about changes in thinking, with these changes in turn associated with changes in culture.

Digital technologies are the cultural tools of today's digitised society. Their role as mediators of human learning is increasingly more complex when one considers the range and scope of computational tools currently available. As mediating tools, they function as intellectual partners with learners in order to enable them to think in ways that otherwise they would not or could not. They amplify, extend and enhance human thinking processes, thus offering a cognitive tool to engage and facilitate cognitive and metacognitive processing (Jonassen, Peck & Wilson, 1999). Jonassen (1996) uses the term 'mindtools' to highlight the power of digital technologies to support knowledge construction and critical thinking. Building on the concept of distributed cognition (Salomon, 1993), he argues that digital technologies should not support learning by attempting to instruct learners but rather should be used as knowledge construction tools that students can learn *with,* not *from.* In this way, learners can be perceived as designers, using the technologies as tools for analysing the world, accessing information, interpreting, organising and constructing their personal knowledge, and representing what they know to others (Jonassen & Reeves, 1996; Jonassen, Peck & Wilson, 1999).

Constructionism is a theory of learning which takes cognisance of the role of cultural tools, while also building on constructivism and sociocultural theories. Papert (1993), who worked with Piaget in the late 1950s and early 1960s, developed this theory of learning based upon Piaget's constructivism. He states:

> *constructionism, my personal reconstruction of constructivism, has as its main feature the fact that it looks more closely than other -isms at the idea of mental construction. It attaches special importance to the role of construction in the world as a support for those in the head, thereby becoming less of a purely mentalist doctrine.* (p. 143)

Papert and Harel (1991, p. 1) further explain how constructionism relates to constructivism with the statement that 'the N word as opposed to the V word – shares constructivism's connotation of learning as 'building knowledge structures'. Learners, consequently, are understood as active builders of their own knowledge and learn with particular effectiveness when they are engaged in constructing personally meaningful artefacts. However, constructionists argue that learning 'happens especially felicitously in a context where the learner is consciously engaged in the construction of a public entity whether it's a sand castle on the beach or a theory of the universe' (ibid, 1991, p. 1).

In this sense, constructionism 'is interested in how learners engage in a conversation with [their own or other people's] artefacts, and how these conversations boost self-directed learning, and ultimately facilitate the construction of new knowledge' (Ackermann, 2001, p. 85).

In our digitised society, from a mathematical perspective, artefacts can include designing and building computer programs, databases, animations or robots. These artefacts are 'objects to think with' (Papert, 1980, p. 12; Turkle, 1995). Through their use, learners are enabled to manipulate and reflect on what they know, and use these reflections to further construct knowledge (Reeves, 1998). They are also a means by which others can become involved in the thinking process. The learner's thinking benefits from interaction with others as the multiple views and discussions that result from such interactions are the greatest source of alternative views needed to stimulate new learning (Von Glasersfeld, 1989). In this way, learners become more engaged in constructing personal and socially-shared understandings of the phenomena they are exploring (Jonassen & Carr, 2000). It follows that the tools and materials used influence the nature of the artefact and therefore the thinking. According to Butler (2007, p. 64), 'There is consequently an interrelatedness of a symbiotic nature that exists between learners, the materials they use and the constructed artefact that they create'. This becomes their 'object to think with'.

Using digital technologies to construct personally meaningful artefacts enables learners to design their own representations of knowledge rather than absorbing representations preconceived by others. As stated by Jonassen and Carr (2000), they 'engage learners in a variety of critical, creative and complex thinking such as evaluating, analyzing, connecting, elaborating, synthesising, imagining, designing, problem-solving and decision making' (p. 168). As such, children not only engage more deeply with content but they can also access powerful mathematical ideas hitherto considered not possible. For example, dynamic geometric software (DGS) programs are tools that can be used to construct and manipulate geometric objects and relations (Battista, 1998; Healy & Hoyles, 2001). Erbas and Aydogan Yenmez (2011) claim that DGS has great potential to impact the teaching and learning of school geometry, particularly if used in a reflection-centred and problem-solving based learning environment. According to Battista (2001), DGS enables children to 'develop rich mental models' which help them 'to reason in increasingly sophisticated ways' (Battista, 2001, p. 118) moving them 'to higher levels of geometric thinking' (Olkun, Sinoplu, & Deryakulu, 2005, p. 11).

To illustrate this, a triangle constructed using DGS will not be a static triangle fixed in space. It can be manipulated to make any desired triangle that fits on the screen, no matter what its shape, size or orientation (Forsythe, 2007). By constructing different triangles and observing the changes in a dynamic manner, the learner is exploring the properties of shape and is not confined to the use of textbooks and commercial sets of 2-D shapes which tend to reinforce visual prototypes (Pengelly, 1999; Frobisher, Frobisher, Orton, & Orton, 2007). A reliance on visual prototypes is characteristic of those operating at Level 0 on the van Hiele geometric reasoning levels. Seventy percent of students leave primary education with a dominant geometric reasoning level of '0' (Battista, 1998) instead of a recommended level two (Van de Walle, Karp, & Bay-Williams, 2010). However, while these tools

have the potential to transform 'mental functioning in fundamental ways' (Chu & Ju, 2010, p. 65), it is imperative that they are used in learning environments that encourage thoughtful reflection (Hannafin et al., 2001; Reynolds & Harel-Caperton, 2011). Consequently, a key role of teachers is to foster the development of a reflective culture in their classrooms (McKenzie, 1998). Thus, constructionism provides a particular perspective on how the use of digital tools impacts children's mathematical thinking and reasoning and promotes the development of their understandings.

A Redeveloped Primary School Mathematics Curriculum

The 1971 mathematics curriculum, *Curaclam na Bunscoile* (Department of Education, 1971), drew heavily on Piagetian ideas, in particular on stage theory. The more recent PSMC (Government of Ireland, 1999) espoused a social constructivist view as evidenced in the emphasis on the social aspects of learning. As discussed in the Introduction to this report, when it was introduced in 1999, the PSMC was well received. While maintaining some important links with the 1971 curriculum, it also drew heavily on Vygotskian ideas about teaching and learning, in that it emphasised the social aspects of mathematics development, the importance of language in acquiring mathematical knowledge, and the key role of the teacher in modelling and supporting children's emerging understanding of mathematics. However, the role of the mathematics curriculum in the minds of teachers is an issue that needs some thought. The issue is how the theoretical underpinnings of the curriculum are commensurate with classroom practice. There is ample evidence that textbooks are used as the main planning tool for the teaching of mathematic in many classrooms (e.g., Eivers et al., 2010; Dunphy, 2009). The design of textbooks, which include pages of repetitive work with barely discernible levels of ascending difficulty (e.g., the repeated practice of addition of two digit numbers without 'carrying' followed later by addition 'with carrying' is at variance with the emphases suggested in the current chapter). Similarly, an understanding of mathematics as largely symbolic and the learning of mathematics as the manipulation of symbols is not coherent with, for instance, the embodied stance of Lakoff and Nunez (2000). An embodied stance is where an idea is expressed or represented physically or concretely. It assumes that young children often communicate and articulate their understandings and ideas by using actions and gestures instead of/as well as words. It might be claimed that the predominance of coloured pictures in current mathematics textbooks has been influenced by 'situated learning' theories, where context is an important basis for learning mathematics. However some of these have been critiqued by Charalambous, Delaney, Hsu and Mesa (2010). Their findings in relation to the addition and subtraction of fractions are that:

> *The Irish textbooks differed from those in the other two countries [Cyprus and Taiwan] in terms of the context around which the worked examples were built. Most worked examples in the Irish texts were set in exclusively mathematical contexts…In the other two countries, worked examples were more often embedded in 'real-world' contexts…Irish textbooks had the greatest number of 'completed' worked examples…all Irish worked examples explicitly illustrated the steps to be followed when completing procedures.* (p. 135)

The above authors argued that there is a need to examine textbooks in order to understand differences in teaching approaches and achievement in different countries. The relevance of this for the mathematics curriculum for children aged 3–8 is that a redeveloped curriculum needs to consider how the range of resources that support pedagogy cohere with the theoretical stance of the curriculum.

Implications for Practice

Table 2.1 below outlines the key implications of the perspectives for learning, teaching curriculum and assessment. Of necessity, these are generalisations. It is important to note that there can be differences in interpretations in relation to the various perspectives, in particular the sociocultural perspectives. This arises from the fact that sometimes theorists who see themselves as located in slightly different places theoretically often use similar concepts and language to articulate their positions (e.g., Ryan & Williams, 2007). This makes their perspectives at times difficult to distinguish.

Table 2.1. Key Implications of Theoretical Perspectives

	Cultural-historical activity theory	Situative
Emphases	▪ The structure of activities ▪ Activity as continually negotiated between participants with the resources of their environments ▪ Tools can be either material or conceptual	▪ The larger systems: includes people, interactions and all the elements of the environment ▪ Practices of the community ▪ Becoming more central in a community's practices
Learning	▪ Learning is the result of everyday practice and processes of meaning-making ▪ An expansive view of learning ▪ Zone of proximal development is a key concept	▪ Learning is a change in participation…about becoming more centrally involved in the practices of the community ▪ Changing forms of participation are part of a process that shape identity formation ▪ Diversity is the expectation: learning more multi-path in nature ▪ Interpretation of artefacts such as symbols and icons is a crucial part of social practices

Table 2.1. Key Implications of Theoretical Perspectives *(continued)*

	Cultural-historical activity theory	Situative
Teaching	▪ Use of tools (for example, technology or symbols) as mediators in activity	▪ The focus is on the group of learners ▪ Dialogical pedagogy of argumentation and discussion designed to support effective conceptual learning ▪ Identification of conceptual obstacles ▪ Scaffolding learning using models ▪ Focus on developing mathematical skills within the context of real-world learning situations ▪ Work with 'rich' mathematical problems e.g. problem-based learning ▪ Foster a community of learners ▪ Foster the development of learner identity ▪ Foster metacognitive awareness ▪ Teach at upper levels of ZPD
Curriculum	▪ Tools can be material or conceptual	▪ Focus on processes with an emergent view on content ▪ Mathematics situated in curriculum tasks which use cultural tools ▪ Mathematical activities must make sense and be part of a child's larger social activity ▪ Models and representations used to solve practical problems
Assessment	▪ Expectations of difference	▪ Assessment of participation in meaningful activities ▪ Diagnosis of errors since these indicate intelligent constructive activity

Table 2.1. Key Implications of Theoretical Perspectives *(continued)*

	Social Constructivist	Constructionism
Emphases	▪ The social context ▪ Interpersonal relations, especially teacher-learner and learner-learner interactions ▪ Negotiation, collaboration, and discussion ▪ The role of language	▪ Constructions in the world as supports for constructions in the head ▪ Tools, media and contexts ▪ Artefacts as objects to think with ▪ Learners construct knowledge particularly well when constructing personally meaningful entities ▪ Learners' reflections and social expression about their work in progress…in a community of practice ▪ Tool use has the potential to transform mental functioning in fundamental ways when combined with thoughtful reflection on the learning process
Learning	▪ Learning is a change in understanding/thinking ▪ Focus on qualitative changes in reasoning ▪ Importance of children reflecting on their work	▪ Learner sets their own learning goals ▪ Emphasises the idea of diversity, recognises that learners can make connections with knowledge in many different ways ▪ Encourages a variety of learning styles and representations of knowledge ▪ Intimate connection between knowledge and activity ▪ Active process that involves individuals asking questions, discussing and solving problems, sharing ideas, thinking critically and exploring and assessing what they know.
Teaching	▪ The focus is on individual learners ▪ Teacher modelling important ▪ Present cognitive challenge ▪ Strategic learning encouraged ▪ Encourage self-regulation of learning	▪ Learning environment fosters discussion and reflection ▪ Learning environment designed to provide opportunities for inquiry-based explorations, collaboration and reflection using a range of computational tools ▪ Foster self-regulation ▪ Foster the development of a reflective culture ▪ Foster culture of collaboration among peers ▪ Reflection/articulation ▪ Foster meta-cognitive awareness ▪ Teacher, or knowledgeable other, participating in the learning process alongside the learner, cueing, prompting, questioning where necessary

Table 2.1. Key Implications of Theoretical Perspectives *(continued)*

	Social Constructivist	Constructionism
Curriculum	▪ Focus on conceptual understanding ▪ Tasks/activities are incremental and build on what children already know ▪ Artefacts used serve to influence thinking	▪ Meaningful, authentic activities that help the learner to construct understandings and develop skills ▪ Long term problems/projects related to the learner's needs and interests ▪ Authentic relevant real-world problems ▪ Learning to learn/thinking about thinking
Assessment	▪ Problem-focused ▪ Authentic tasks focused on a wide range of cognitive behaviours (lower and higher order) ▪ Aimed at eliciting expertise	▪ Encourage learners to make predictions and to constantly reflect on discrepancies between their predicted answers and those found. As they do so they refine their theories and understandings.

In the United States, cognitive science emphases are reflected in many high-profile statements (e.g., NRC 2001, 2005; NCTM, 2000). They are also reflected in the work of prominent early childhood mathematics educators (e.g., Clements, Sarama & DiBiase, 2004). Meanwhile, in countries such as Australia there has been a movement amongst mathematics educators and in curriculum policy towards socioculturally-oriented approaches to teaching, learning, assessment and curriculum. See for instance Conway and Sloane's (2005) account of changes in assessment practices in Victoria and New South Wales. See also Perry and Dockett's (2008) articulation of a socioculturally oriented mathematics curriculum at preschool level, first presented as early as 2002.

The PSMC (Government of Ireland, 1999) can be seen as having a socio-constructivist orientation which had its roots in Piagetian/radical constructivism, though there are also some adherences to a Vygotskian perspective. Social constructivism has two formulations, one with its roots in Piagetian/radical constructivism, and the other with its roots in Vygotskian theory (Ernest 2010, p. 54). We consider this distinction helpful in considering how the theoretical orientation of a redeveloped curriculum for the mathematics education of children aged 3–8 years might be distinguished from that of the 1999 PSMC. A new iteration of the curriculum which takes account of the sociocultural perspectives described above would be much more firmly rooted in recent theories developed from a Vygotskian base and which emphasise children's participation in mathematics, their identity as mathematics learners, and their interactions in communities of learners.

Conclusion

In early childhood mathematics education sociocultural/cultural-historical theories are of particular importance, given their capacity and usefulness for explaining early learning and the role of cultural and social influences in learning. Recent versions of constructivism help to explain the mechanisms of learning and these are central to a comprehensive theory of early mathematics learning. The insights afforded by considering the cultural and social dimensions of the learning situation, including cultural tools and media, explain what children learn, why they learn in particular circumstances and how they learn. They also indicate clearly how early mathematical learning and development can best be supported. An explanatory framework recognising the role of internal processes, but foregrounding the fact that mathematics learning and development are dependent on children's active participation in social and cultural experiences, provides the basis for a powerful theoretical framework for mathematics education for children aged 3–8 years. Important too we feel are the insights offered by the Realistic Mathematics Education (RME) approach. However, we have left our discussion of that until Chapter 5 (Section: *Developing Children's Mathematical Thinking: Three Approaches*) since RME is an approach to mathematics education, rather than a 'grand theory' of learning.

The key messages arising from this chapter are that

- Cognitive and sociocultural perspectives provide different lenses with which to view mathematics learning and the pedagogy that can support it. Cognitive perspectives are helpful in focusing on individual learners, while sociocultural perspectives are appropriate when focusing on, for example, pedagogy.
- Sociocultural perspectives, cognitivist perspectives and a constructionism perspective each offer insights which can enrich our understanding of issues related to the revision of the curriculum. They do so by providing key pointers to each of the elements of learning, teaching, curriculum and assessment. Used together, they can help in envisaging a new iteration of the PSMC.
- Learning mathematics is an active process which involves meaning making, the development of understanding, the ability to participate in increasingly skilled ways in communities of learners, and engagement in mathematization and the development of a mathematical identity.
- The proactive role of the teacher must be seen to involve the creation of a zone of proximal development, the provision of scaffolding for learning, and the co-construction of meaning with the child based on awareness and understanding of the child's perspective. It also involves a dialogical pedagogy of argumentation and discussion.

CHAPTER 3

Language, Communication and Mathematics

Language plays a critical role in developing young children's mathematical thinking (e.g., Ellerton, Clarkson & Clements, 2000; Whitin & Whitin, 2003). Talking about mathematical thinking and engaging in reasoning, justifying and argumentation are central to mathematics education for all children aged 3–8 years (Ginsburg, 2009a). According to the NRC report:

> *Children must learn to describe their thinking (reasoning) and the patterns they see, and they must learn to use the language of mathematical objects, situations and notation. Children's informal mathematical experiences, problem solving, explorations, and language provide bases for understanding and using this formal mathematical language and notation.* (2009 p. 43)

In his seminal work on mathematics register, Haliday (1978) argues that acquiring mathematics involves learning not just the vocabulary of mathematics, but also the styles of meaning, modes of argument, and methods of thinking mathematically. Similarly, Schleppegrell (2010) calls on educators to view mathematics as discourse. In this view planned activities provide opportunities to engage learners in such discourse, without losing a focus on the underlying mathematics. This perspective is consistent with sociocultural theories of mathematics learning which see children being enculturated into mathematics through social activity and discourse (see Chapter 2, Section: *Sociocultural Perspectives*). Perry and Dockett (2002) emphasise the value in allowing young children to use their own symbols and their own names for mathematical entities in the early stages of learning mathematics, followed by a gradual shift to more formal systems. They also draw attention to the challenges facing young children in settings where the discourse of mathematics involves a language that is different to the language of the home.

The term 'math talk' is often used to describe the language interactions that occur when children are supported in talking about their mathematical thinking, including their formal and informal representations of mathematical ideas and symbols. Indeed, the NRC report (2009) notes that a 'math-talk learning community', in which all children have opportunities to describe their thinking, has the potential to improve children's mathematical language and their general language levels. It also points to the importance of children using language to make connections across different domains of mathematics, and across mathematics, other learning areas, and everyday life.

The importance of oral language in developing mathematical understanding is recognised in policy statements and curriculum documents. For example, the NAEYC/NCTM (2002/2010) *Position Statement on Early Childhood Mathematics* (3–6 years) includes as a recommendation the active introduction of 'mathematical concepts, methods and language through a range of appropriate experiences and strategies' (p. 9), while taking children's cultural background and language into

consideration. The same report notes the many opportunities that can arise to integrate mathematics with other learning activities (e.g., storytelling) which can support children in learning mathematical vocabulary. *Aistear* (NCCA, 2009a) includes, as one of the key aims of its *Communicating* theme, a broadening of children's understanding of the world by making sense of experiences through language, including mathematical language.

This chapter examines the role of language in learning mathematics. First, it looks at the relevance of language for learning different aspects of mathematics and the research that supports the use of mathematical language in children's homes, in the preschool and in primary school. Second, it looks at theories of communication in mathematics learning and links them to broader theoretical frameworks for learning mathematics that were considered in *Chapter 2*. Third, it describes the development of children's mathematical vocabulary in the context of broader conceptual development. Fourth, it considers groups who may struggle with language in general, and therefore may experience additional challenges in bridging the gap between informal and more formal mathematical ideas.

The Role of Language in Developing Mathematical Knowledge

There is a complex relationship between language development and growth in mathematical thinking. Even before they acquire language it seems that infants in their first year may be aware of changes in the numbers of items in small sets (e.g., Feigenson & Carey, 2005) and can discriminate between larger sets of items where the proportional difference is large (e.g., 8 vs. 16 items) (Brannon, Abbott & Lutz, 2004). In these early stages, there is a complex relationship between representation of number, and representation of associated variables such as area, size and arrangement of items. Moreover, such early number representations may work independently of the language system (e.g., Gelman & Butterworth, 2005). According to Zur and Gelman (2004), 3-year-olds can use basic number concepts to predict and check the results of additions and subtractions to sets of up to five items, even if they are unable to produce such sets by counting.

There is some disagreement among researchers as to when children integrate their number word knowledge (e.g., counting) with their non-verbal number systems. Carey (2004) suggests that language factors (including knowledge of plurals) can 'bootstrap' number development as they combine with earlier non-verbal representations of number, to provide a new and comprehensive number system. For example, children's knowledge of number word sequence, which may have been acquired initially without numerical meaning, combines with their representations of small sets of items. This combination is seen as providing a basis for symbolic representation of number.

Others (e.g., Rips, Asmuth & Bloomfield, 2008) argue that knowledge of the number word sequence is not sufficient to support conceptual development, and that it is only at a much later stage – called 'advanced counting' – that children can construct the next number term from any number in the sequence, based on the correspondence between the structure of the number sequence and the properties of natural numbers. This is evident in a study by Sarnecka and Carey (2008) in which

children aged 2 years and 10 months to 4 years and 3 months were the subjects. While almost all children could produce the number sequence to 10, conceptual understanding varied considerably, with 40% of children showing no understanding that going forward in the number sequence corresponds to adding, and going back to subtracting.

Taken together, such studies suggest a need for early childhood educators to support young children in establishing a conceptual link between language (in this case, the number sequence) and understanding of number. According to Donlan (2009), the process of integrating procedures and concepts (e.g., rote counting and underlying principles of counting) is important.

Adult Support

As young children grow and develop so too does their familiarity with and use of language. Everyday situations both support and encourage children's use of mathematically-related language, especially where these involve interactions with adults.

There is evidence that the mathematical language used by adults in preschool settings can have an impact on children's mathematical knowledge. In a study involving 26 preschool teachers and their children, Klibanoff et al. (2006) recorded instructional time, including circle time, over a seven-month period for one hour per month in each class. Although few teachers led planned mathematics lessons during the recorded observations, many incorporated mathematical inputs in their speech. Children were pre- and post-tested on mathematical knowledge. Children in settings in which teachers used many instances of math talk were more likely to improve over the course of the study than children in settings in which less mathematical language was used. An interesting outcome of the study was the wide range of mathematical inputs across settings, ranging from 1 to 104 instances of mathematical utterances, with an average of 28. Forty-eight percent of all inputs were references to cardinality, while inputs relating to equivalence, non-equivalence, ordering, calculation and placeholding were much less common. This outcome of this study is consistent with the work of Dickinson and Tabors (2001), whose research with preschoolers showed that, during large-group activities, more frequent use of teachers' explanatory talk and use of cognitively challenging vocabulary were associated with better learning outcomes for children.

Familiarity with spatial language is particularly important in learning and retaining spatial concepts. Gentner (2003) found that children who heard specific spatial labels during a laboratory experiment that involved hiding objects ('I'm putting this on/in/under the box') were better able to find the objects than children who heard a general reference to location ('I'm putting this here'). Moreover, this was true even two days later, without further exposure to the spatial language (Loewenstein & Gentner, 2005). Szechter and Liben (2004) observed parents and children in the lab as they read a children's book with spatial-graphic content. They found an association between the frequency with which parents drew children's attention to spatial-graphic content during book reading (e.g., 'The Rooster is really tiny now') and children's performance on spatial-graphic comprehension tasks.

Levine et al. (2012) examined how parents use spatial language during puzzle play in a study in which parent-child pairs were observed for an hour during naturalistic interactions every 4 months from 26–46 months. Children who were observed playing with puzzles performed better on a mental rotation task at 54 months, after controlling for parent education, income and overall parent word types. Further, among those who engaged in playing puzzles during observations, those who played more puzzles did better. Although the frequency of puzzle play did not differ for boys and girls, the quality of puzzle play (a composite of puzzle difficulty, parent engagement, and parent spatial language) was higher for boys than for girls. In interpreting this, Levine et al. (2012) raised the possibility that girls might benefit from more complex puzzles. There is also evidence that higher amounts of parent spatial language occur during guided block play in which there is a goal than during free play with blocks (Shallcross et al., 2008). Thus, it is possible that spatial activities, spatial language, or both promote the development of spatial skills, such as block building and mental rotation.

Language is one domain-general cognitive skill on which young children may vary. Others include memory, visual-spatial skills, and executive functions (Mazzocco, 2009), though none of these are independent of one another and, like language, they are associated with learning difficulties in mathematics.

The Nature and Scope of Mathematical Discourse

Language plays as important a role in mathematics learning as in other school subjects (Schleppegrell, 2010). While teaching the vocabulary of mathematics to young children is important (e.g., Neuman, Newman & Dwyer, 2011), research has gone beyond the word level in identifying and describing the language challenges of mathematics. Haliday (1978), for example, refers to a mathematics register – 'the meanings that belong to the language of mathematics' (p. 79). In this sense, learning the language of mathematics does not entail just learning new words, but also learning new 'styles of meaning and modes of argument…and of combining existing elements into new combinations' (pp. 195–196). Hence, while activities such as counting and measuring may well draw on everyday language, children learning mathematics need to use language in new ways to serve new functions. According to Schleppegrell (2010), the concept of a mathematical register draws attention to the ways in which mathematical knowledge is different from knowledge of other academic subjects. She argues that learners need to be able to use language to participate effectively in 'ways of knowing that are particular to mathematics' (p. 79). Hence, if we view mathematics as discourse, we need to identify ways of apprenticing children into particular ways of doing mathematics in particular discursive contexts. Pimm (1991) argues that children in school are attempting to acquire communicative competence in the mathematical register, and that classroom activities should be carefully examined from this perspective in order to see what opportunities they offer for children's language learning. Silver and Smith (1996) point out that, in developing and using language in mathematics, it is important that mathematics does not get lost and that discourse focuses on 'worthwhile tasks that engage students in thinking and reasoning about important mathematical ideas' (p. 24).

A number of theories reviewed in *Chapter 2* under the broad umbrellas of constructivist/cognitive and sociocultural can be drawn on to explain the relationship between language and mathematics:

- Cognitive theories, which have their origins in the work of Piaget, focus on the individual child's construction of internal representations or structures. According to Cobb and Yackel (1996), constructivist perspectives can be characterised as interpretive, since knowledge is actively constructed by children in interaction with their environment. Constructivists focus on the way children talk about mathematics to investigate their development of mathematical knowledge.
- Sociocultural theories focus on discursive practices and the interaction of children. They draw on Vygotskian frameworks that stress the interaction between language and cognition and highlight the social dimension of language and the role of communication and participation. In sociocultural terms, children are enculturated into mathematics through social and discursive activity.

Other researchers (e.g., Cobb, Yackel & McClain, 2000; Gutiérrez, Sengupta-Irving & Dieckmann, 2010) have built on cognitive and sociocultural theories to view language as a tool for thinking, interpreting, constructing knowledge and developing mathematical ideas. In this view, oral language is one of a range of resources that also include written language, gesture, symbols, equations, graphs and other visual representations. Hence, all of these need to be taken into account in interpreting how children construct meaning during mathematical activities. Children coming from different backgrounds and contexts may be positioned in different ways to use these resources. According to Schleppegrell (2010), differences should be acknowledged and viewed as resources in the mathematics classroom if the focus is on meaning, and if teachers are able to draw on different perspectives.

Sfard (2007) makes a useful distinction between language and discourse when she identifies language as a tool and discourse as an activity in which the tool (one of several) is used or mediates. For Sfard, knowing mathematics is synonymous with the ability to participate in mathematical discourse. Hence, learning is a special type of social interaction aimed at modification of other social interactions. An implication of this is that teachers can help modify children's everyday discourse into a more mathematical discourse. Interestingly, Gutiérrez et al. (2010) point out that Sfard's communicational approach to mathematics does not imply that children must first encounter a mathematical idea, use it, and then formalise it later into mathematical conventions (the 'learning with understanding' approach). Instead, Sfard proposes that an existing discourse of mathematics (e.g., thinking about big numbers or infinity) can be used to initiate children into a discourse of new objects.

Sfard's work can also help to clarify the distinction between everyday (colloquial or primary) discourse and literate (scientific or secondary) discourse. Sfard (2001) argues that everyday discourse does not naturally evolve into scientific (e.g., mathematical) discourse. This is because mathematical discourses are mediated by symbolic artefacts designed to communicate specific conceptual understandings of quantities (that is, symbolic mediation is a key characteristic of mathematical discourse). Since such discourse is often not a part of children's everyday discourse, secondary discourse requires explicit teaching (Sfard & Cole, 2003, cited in Gutiérrez et al., 2010).

Gutiérrez et al. (2010) point out the importance of viewing everyday discourse and scientific (here, mathematical) discourse relationally. This implies that, rather than everyday discourse being viewed as a pre-requisite for mathematical discourse, mathematical discourse can be viewed as arising from (and feeding back into) everyday discourse. The two discourse types can be viewed as operating side-by-side, each being invoked in different circumstances depending on the context involved. An implication of this perspective is that children's general language skills can develop as a result of participating in mathematical discourse.

O'Halloran (2005) has focused on the characteristics of successful mathematics discourse as it relates to other available tools: (i) the meaning potential of language, symbolism and visual images are accessed; (ii) the discourse, grammatical and display systems of each resource function integratively; and (iii) meaning expansions occur when the discourse shifts from one resource to another (p. 204).

Establishing a Math-Talk Culture

NicMhuirí (2011) points to some of the differences between the discourse of traditional mathematics lessons, and the discourse of mathematics lessons that seem to engage children in mathematical discourse. While the former are often characterised by 'repeated iterations of lower-level questions' (p. 320) or the IRF (invitation-response-feedback/evaluation) pattern, and dominated by teacher talk, the latter can include 'patterns of dialogue that involve making conjectures, and examining and justifying one's own mathematical thinking and the mathematical thinking of others' (p. 320). Although NicMhuirí's analysis of mathematics lessons focused on third to sixth classes, her outcomes may have implications for mathematics teaching more generally. In particular, she identifies less helpful patterns of discourse where

- teacher intervention focuses on the solution provided by children rather than their mathematical thinking
- there are lengthy teacher explanations between questions/dialogue
- learners are prompted to arrive at a correct answer, with the teacher sometimes taking on the cognitively-demanding aspects of the task, and, on other occasions, focusing children's attention on critical aspects of the problem, even if the children were expected to solve the problem on their own.

Although the pattern of interactions in the lessons analysed by NicMhuirí may have been justified on the grounds that they keep the lesson moving along towards an end-goal, important opportunities for engaging in mathematical dialogue, including mathematical reasoning, may be overlooked. This and similar work (e.g., Dooley, 2011) point to a need to support teachers to reflect on their classroom dialogue, and provide children with more opportunities to engage in mathematical thinking, along the lines described earlier. Indeed, the relative difficulty that children in Ireland, including those in second class, encounter with solving mathematics problems (see *Introduction*)

point to the urgency of promoting more interactive mathematical discourse in learning settings. Others (e.g., Hufferd-Ackles et al., 2004) provide a framework for establishing and developing math-talk learning communities in learning contexts.

It also seems relevant, in the context of supporting mathematical discourse in early learning settings, to draw attention to more general strategies for language development (e.g., Dooley, 2011; Shiel et al., 2012) that teachers can implement including

- following the child's lead
- mapping language to the child's focus of attention
- cueing/prompting and inviting further comment
- extending the topic by providing further comment
- use of repetition, recasts and expansions
- modelling correct use of vocabulary in sentences
- use of topic elaboration.

NicMhuirí's work also highlights the importance of teachers engaging children in discussing and solving problems among themselves. This is consistent with sociocultural theories of learning that emphasise the role of language in acquiring knowledge in social communities, and with more general theories of learning mathematics that emphasise the role of argumentation (e.g., Perry & Dockett, 2008).

Learning Mathematical Vocabulary

In earlier sections of this chapter, we noted the importance of vocabulary in establishing bridges between young children's early sense of number and spatial sense, and their later mathematics learning. While mathematical vocabulary can be taught in formal or semi-formal settings such as maths classes, it can also be taught informally. As noted above, there is research evidence linking the frequency of adults' use of mathematical vocabulary in informal activities such as playing with bricks or solving a puzzle/jigsaw that can impact on children's mathematical learning.

Efforts have been made to specify the mathematical vocabulary that young children should learn. For example, in the current PSMC, specific mathematical vocabulary which should be addressed is highlighted in the content objectives. In matching equivalent and non-equivalent sets, children should be supported in learning terms such as *more than, less than, enough* and *as many as*. In developing spatial awareness, such terms as *above*, *below, near, far, right* and *left* are identified as a focus of instruction. In the United Kingdom, in support of the *National Mathematics Strategy*, the UK Department for Education and Employment (DfEE, 2000) issued a booklet for teachers that

specified the range of vocabulary to be taught at each class level from reception (age 5 years) to year 6. In reception year, the mathematical areas under which vocabulary items are grouped include: counting and recognising numbers; adding and subtracting; solving problems; measures; shape and space; instructions and general. Importantly, the vocabulary booklet notes that key terms should, where possible, be taught in context, and instruction should be supported by the use of relevant real objects, mathematical apparatus, pictures and diagrams. The use by teachers of questions (both open-ended and closed) that enable children to use new vocabulary is stressed, and teachers are urged to be sensitive to the possibility that some vocabulary terms may be well understood by children in non-mathematical contexts or everyday language, but not in contexts where more precise mathematical understanding is important. In addition to targeted teaching of key vocabulary, opportunities should also be sought to support children's learning and the use of mathematical vocabulary in a range of contexts including play, mathematics lessons (e.g., when solving problems), and other learning areas.

Some researchers working with socio-economically at-risk preschool or kindergarten children (e.g., Neuman, Newman & Dwyer, 2011) have drawn attention to how such children often lack the conceptual knowledge required to understand mathematical discourse, and may need a more intensive approach to vocabulary development, compared with children who are not at risk. They report on a year-long programme administered to 3- to 4-year-olds in US Head Start classrooms that focused on word knowledge and conceptual development through taxonomic categorisation (categorising words) and embedded multimedia. Children in the programme, which covered aspects of health education (50 words) and living things (80 words) as well as mathematics (geometric shapes and number) (50 words), outperformed their counterparts in a control group on a range of outcome measures including domain-specific knowledge. Moreover, gains in word and categorical knowledge were sustained six months later. The authors interpreted the findings as suggesting that teaching words within taxonomic categories 'may act as a bootstrap for self-learning and inference generation'. The programme made a distinction between the concepts to be taught (e.g., some geometric shapes have corners, and some do not) and the target vocabulary words (e.g., specific shapes), with an instructional emphasis on both.

Variation in Language Skills and Impact on Mathematics

A number of groups are known to struggle with general language acquisition, including children living in disadvantaged circumstances, children who speak a language other than the language of instruction at home, and children who have a language impairment (see also the discussion in Chapter 6, Section: *Immersion Settings*).

We know that children living with disadvantage do not lack fundamental mathematical ability and that these children demonstrate few if any differences in the everyday mathematics they use in free play (e.g., Ginsburg et al., 2008). Familiarity with mathematics language is generally recognised as a key issue that must be addressed in early childhood mathematics education (e.g., Ginsburg, 2009a;

Hughes 1986). Mathematical language includes vocabulary, but just as crucial are language skills that enable the communication of mathematical thinking. The urgency to ensure that children living with disadvantage have adequate language experiences around mathematics is also emphasised by Perry and Dockett (2008) who argue that, 'without sufficient language to communicate the ideas being developed, children will have the opportunities for mathematical development seriously curtailed' (p. 93). Furthermore, they contend that the development of mathematical language, especially among non-English speakers, is particularly problematic because of mathematics' specialised vocabulary and because common words have specialised meanings.

In the United States, approximately 7% of children have specific language impairment (SLI), and while there is considerable variation within this group, many experience difficulty in learning the number-word sequence (Donlan, 2009). In one study, 5-year-olds with SLI were able to recite the number sequence up to 6, while their non-SLI counterparts reached 20 (Fazio, 1994). However, contrary to expectations, the SLI group showed a good understanding of the logical principles in object counting, including the principle that the final count word indicates the value of the set. When Fazio retested the children with SLI at 2-year intervals, they struggled on measures of basic calculation (Fazio, 1996, 1999).

A similar pattern of procedural weakness and conceptual strength emerged in a study of 7-year-olds with SLI. Forty percent of the group were unable to count to 20, whereas just 4% of typically developing 5- to 6-year-olds were unable to do so. Again, the performance of the children with SLI on a test of understanding of arithmetic principles was similar to typically-developing peers (Donlan, Cowan, Newton & Lloyd, 2007).

Nevertheless, Donlan (2009) warns that it is incorrect to accept that the effects of language difficulties on mathematical development are delimited in a clear way, with non-verbal number processes relatively unaffected. He points to a need for additional research that highlights how factors underlying SLI might impact on SLI children's performance on tasks of enumeration and calculation.

Conclusion

Language plays a key role in the development of children's mathematical thinking. Cognitive/constructivist and sociocultural theories of learning (see *Chapter 2*) support a strong focus on the use of language to acquire mathematical knowledge, and adults – whether parents, carers or teachers – are seen as key agents in supporting children's development of mathematical language across a range of informal and more formal contexts. While some of the mathematical language used in preschool and early school settings will be informal and will arise from children's participation in everyday activities (e.g., counting the number of children in a group, matching coats to children), other instances of language use will be planned around specific activities such as block building, solving puzzles/jigsaws, shopping or using mathematical software. These provide significant opportunities to introduce relevant mathematical vocabulary, engage children in using mathematical language

through asking open or closed questions, paraphrasing or extending children's responses, and encouraging them to explain their thinking. Most importantly, children should be provided with opportunities to engage in mathematical talk with other children.

The key messages arising from this chapter are as follows:

- Cognitive/constructivist and sociocultural perspectives on learning emphasise the key role of language and dialogue in supporting young children's mathematical development. Emerging learning theories point to the importance of mathematical discourse as a tool to learn mathematics.
- In addition to introducing young children to mathematical vocabulary, it is important to engage them in 'math talk': conversations about their mathematical thinking and reasoning.
- Research indicates an association between the quality and frequency of mathematical language used by carers, parents and teachers as they interact with young children, and children's development in important aspects of mathematics. This highlights the importance of adults modelling mathematical language and encouraging young children to use such language as they engage in dialogical reasoning. Children's conversations among themselves about mathematical ideas can also support their development of mathematical knowledge.
- Children at risk of mathematical difficulties may need additional, intensive support to develop language and engage in mathematical discourse. In this context, extensive care and attention should be given to the language element of the learning and teaching of mathematics and extra supports should be provided in these contexts.

CHAPTER 4

Defining Goals

In *Chapter 2* we saw how theories of mathematics learning have moved away from seeing learning as acquisition of knowledge towards seeing learning as the understanding of the practice of doing mathematics. This change in perspective implies the need for new learning goals for mathematics education. These new goals need to emphasise understanding and thinking as well as skills and facts. The specification of goals is an issue that is closely linked to pedagogy since different practices support different goals (Gresalfi & Lester, 2009). Awareness of the need to balance process and content goals is evident in a recent characterisation of early mathematics education in the United States (e.g., NRC, 2009). This focus is encapsulated in the following statement from Clements, Sarama and DiBiase (2004):

> *As important as mathematical content are general mathematical processes such as problem solving, reasoning and proof, communication, connections, and representation; specific mathematical processes such as organising information, patterning, and composing; and habits of mind such as curiosity, imagination, inventiveness, persistence, willingness to experiment and sensitivity to patterns.* (p. 3)

In this chapter we examine approaches to the specification of goals for early childhood mathematics education more closely. In identifying goals for young children's mathematics learning, commentators take different approaches and may choose to foreground particular goals. This is largely dependent on theoretical orientation, conceptions of mathematics, context and the age-range that they focus on. In this chapter we discuss one overarching framework related to higher-order thinking. We discuss two different approaches to the specification of goals for early childhood mathematics, one from a sociocultural perspective and one from a cognitive perspective. We consider how they deal with the content/process issue and we compare the approaches with that used for the specification of goals in the 1999 PSMC. We consider the implication for the structure of curriculum materials.

A Coherent Curriculum

The curriculum should have continuity from early childhood through all phases of education. One way of doing this and of mitigating discontinuities in mathematics learning is by having agreed goals, the nature of which can become more subject specific as children grow older (e.g., Pound, 1999). In Ireland, revised goals for mathematics education will need to build on the broad learning goals related to each of the themes of *Aistear* (*Well-being, Identity and Belonging, Communicating*

and *Exploring and Thinking*). These themes provide the foundations on which subsequent mathematics-specific learning is based. In addition, the revised goals also need to be consistent with the goals of *Project Maths*.

Essentially, underlying views of mathematics, of knowledge and of learning are what determine the nature of the goals that are specified in the curriculum. In curricula, all of the elements, including theoretical orientation and goals, must align. In the final analysis, considerations related to early learning and the relative weight given to cognitive and social processes are key issues that serve to guide the specification and presentation of goals. In the section which follows, we begin by describing goals for mathematics education which are over-arching and expressed at a very general level. Then we describe two different approaches to thinking about skills and concepts. First we discuss Perry and Dockett's (2008) concept of *powerful mathematical ideas* as a unifying approach to emphasising both processes and content. Then we consider Sarama and Clements' (2009) goals, which they refer to as *big ideas* that they present as content-oriented goals, while stressing processes as implicit in these goals.

Specifying Goals

Overarching Goals for Mathematics Education

Higher-Order Thinking

Taking an international perspective, Cai and Howson (2013) argue that there are commonly accepted learning goals in school mathematics – the development of knowledge and skills along with an emerging emphasis on the development of higher-order thinking skills. In the absence of commonly accepted definitions, they utilise Resnick's (1987) characterisation of higher-order thinking as non-algorithmic, complex, with multiple solutions; involving nuanced judgement, application of multiple criteria, uncertainty, self-regulation, imposing meaning; and effortful. Cai and Howson (2013) draw attention to the flexibility and self-monitoring (meta-cognition) that these skills involve. These authors also emphasise the ability to work together with others as essential to the development of higher-order skills.

Specific mathematical processes employ higher-order skills. While some of these skills may appear to be very abstract in terms of young children's mathematical thinking, all of them have their genesis in early childhood. For instance, Australian researchers Perry and Dockett (2008) point out that argumentation is now seen as central to the mathematics development of young children. Citing Krummheuer (1995, p. 229), they define argumentation as a 'social phenomenon, when co-operating individuals [try to] adjust their intentions and interpretations by verbally presenting the rationale of their actions.' They are concerned that there is recognition of what argumentation might look like in young children.

Engaging with Powerful Mathematical Ideas

Taking a strongly process-oriented approach and from a vantage point of sociocultural theory, Perry and Dockett (2008) propose a list of powerful mathematical ideas, to which they believe most young children have access (see Table 4.1). Again, an example of a 'powerful idea' is argumentation. These ideas combine processes and content, with processes foregrounded. In their view, knowledge and skills are developed through engaging in mathematical processes. They identify four important issues for the development of knowledge and skills, particularly at the school level: models and modeling, language, technology, and assessment. In their judgement, these key processes, when well-conceived, understood and promoted by teachers, can serve as critical drivers in the development of the powerful mathematical ideas that children need to understand. They emphasise children's purposeful use of mathematics in their everyday lives in prior-to-school settings and in out-of-school settings. They focus on the centrality of using children's understandings built up through engagement in everyday activity as a basis for learning and teaching mathematics in the range of early education settings.

Exploring the Big Ideas in Mathematics Learning

Mathematics educators, especially in the US, make frequent reference to the need for teachers to understand the 'big ideas' in young children's mathematics learning and use them to connect ideas in mathematics (NCTM, 2000). Baroody, Purpura and Reid (2012, p. 164) explain that these ideas interconnect various concepts and procedures within a domain and across domains. They represent 'big leaps' in the development of children's reasoning and can be seen, according to Fosnot and Dolk (2001, p. 11), as both 'deeply connected to the structures of mathematics…[and] also characteristic of shifts in learners' reasoning'.

However, a definite list as to what exactly these ideas might be is more difficult to ascertain. Some examples of what different commentators understand as big ideas are to be found in the literature. For instance, enumeration (determining a set's numerical value) is underpinned by a set of mathematical ideas such as cardinality (e.g., Ginsburg, 2009b). Unitising underlies the understanding of place value (e.g., Fosnot & Dolk, 2001).

From an early childhood perspective, Sarama & Clements (2009) define their big ideas in mathematics as

> *overarching clusters and concepts and skills that are mathematically central and coherent, consistent with children's thinking, and generative of future learning. This organisation reflects the idea that children's early competencies are organised around several large conceptual domains.* (pp. 16–17)

These authors appear to conflate the idea of goals with that of 'big ideas' (e.g., Smith-Chant, 2010a). They suggest that in early childhood mathematics there are about twelve big ideas that need to be built up incrementally over time (see Table 4.1). They identify at least one accompanying big idea for verbal and object counting: counting can be used to find out how many are in a collection. Their big idea of composition and decomposition of shape has at least one associated big idea: geometric shapes can be described, analysed, transformed and composed and decomposed into other shapes. Sarama and Clements' work on learning trajectories (see *Chapter 5*) is built on the big ideas, or goals, they identify as essential for the learning and teaching of early mathematics. The explication of each of their goals is based on several decades of work in the cognitive sciences which they synthesised and presented in the form of developmental progressions. Clements and Sarama (2009a, p. 6) stress that their goals focus on far more than facts and ideas, and that processes and attitudes are important in each goal. However, processes are not explicit in their specification of goals. This issue of how processes are presented and integrated with skills and content is one that is critical in terms of the presentation of the redeveloped curriculum.

Table 4.1. Specifying Goals: Different Approaches

Perry and Dockett (2008)	**Sarama and Clements (2009)**	**Primary School Mathematics Curriculum (1999)**
Powerful mathematical ideas (a sociocultural perspective)	Big ideas (a cognitivist perspective)	A socioconstructivist/ sociocultural perspective
▪ Mathematization ▪ Connections ▪ Argumentation ▪ Number sense and mental computation ▪ Algebraic reasoning ▪ Spatial and geometric thinking ▪ Data and probability sense	▪ Counting ▪ Ordering numbers ▪ Recognising number and subitising ▪ Knowing different combinations of numbers ▪ Adding and subtracting ▪ Multiplying and dividing ▪ Measuring ▪ Recognising geometric shapes ▪ Composing geometric shapes ▪ Comparing geometric shapes ▪ Spatial sense and motions ▪ Patterning and early algebra	▪ Applying and problem-solving ▪ Understanding and recalling ▪ Communicating and expressing ▪ Integrating and connecting ▪ Reasoning ▪ Implementing ▪ Early mathematical activities ▪ Number ▪ Algebra ▪ Shape and space ▪ Measures ▪ Data

The Structure of Curriculum Materials

A particular issue in relation to curriculum implementation has been a widely acknowledged difficulty in the integration of processes, skills and content, with teachers placing greater emphasis on procedural aspects of mathematics than on broader educational goals (Anderson, White, & Sullivan, 2005; Eivers et al., 2010; Handal & Herrington, 2003; Ross, McDougall, Hogaboam-Gray, & LeSage, 2003). Numerous factors have been identified to explain the mismatch between 'intended', 'enacted' and 'attained' curricula (Cuban, 1993; Robitaille & Garden, 1989) – most particularly teacher beliefs (Anderson et al., 2005; Handal & Herrington, 2003). However, some attention has been given recently to the objective structure of curriculum materials (Herbel-Eisenmann, 2007). While acknowledging the complex and multifaceted nature of the teacher-curriculum relationship, Remillard (2005) urges curriculum developers to take account of this relationship in the design of materials:

> *…[C]urriculum materials have a number of characteristics beyond the specific mathematical content and pedagogy represented in the text. These characteristics include the look and voice of the text and its subjective scheme or how it is perceived. It is critical that curriculum developers pay careful attention to the multiple ways that their materials communicate with the teacher. They must consider how they are addressing the teacher through the design of their materials, how they expect the teacher to respond to their suggestions, and how they represent what it means to use their resource.* (p. 240)

Comparing the Perry and Dockett specification of goals with the Sarama and Clements specification, we can see that, while the former foregrounds processes but includes content areas, the latter appears to focus on content and sees processes as implicit. The question is whether one or the other approach is preferable in terms of key organisers in the redeveloped maths curriculum. There is also the issue of which presentation best promotes continuity of experiences and pedagogy in different settings. Advantages and disadvantages can be identified with both approaches.

The Perry and Dockett specification foregrounds processes. This is consistent with their sociocultural perspective on learning. They lead with mathematization, a process which can actually be seen as content since as children explore a mathematical idea they are involved in the content of mathematics (e.g., Fosnot & Dolk, 2001). There are two readily identifiable arguments for a specification such as this one. The first relates to coherence – among the conditions that Schoenfeld (2002) identifies for high quality mathematics teaching is the development of 'coherent curricula rather than disconnected sets of activities' (p. 9). Given the sociocultural/situative view of mathematics, of mathematics education and of pedagogy espoused in previous chapters of this report, a specification with a strong focus on process makes for a greater degree of coherence. The second relates to how the curriculum presents to teachers. The Perry and Dockett list is a balanced one with processes listed before content, thus signalling to educators a revised emphasis.

In Ireland, the PSMC is presented in two distinct sections. In many respects, this curriculum is quite detailed. It includes a skills development section which describes the skills that children should acquire as they develop mathematically. It also includes a number of strands which outline content that is to be included in the mathematics programme at each level. Each strand includes a number of strand units. These are further broken down, mainly with reference content objectives, with a number of these related to each strand unit. However, research now suggests an alternative approach to breaking down the goals into large numbers of objectives. This involves a specification of key mathematical ideas and critical transitions.

Breaking Down the Goals: Critical Transitions within Mathematical Domains

Goals for mathematics learning can be developed at different levels of detail. Above we saw that Sarama and Clements (2009) appear to conflate the idea of goals with that of 'big ideas'. However they also allude to accompanying ideas which indicate key insights in relation to children's understanding of the goal or big idea. For instance they identify the notion that counting can be used to identify how many are in a collection, as a key insight in relation to verbal and object counting. Simon and Tzur (2004) also reference cardinality, but refer to it as a key developmental understanding (KDU). The big leaps and shifts in reasoning as described by Fosnot and Dolk (2001) and referenced earlier in the chapter appear to us to be analogous to KDUs.

Simon considers that KDUs are essential in that they identify 'critical transitions that are essential for children's understanding of a particular concept or domain' (p. 360). Furthermore, he argues that they provide the basis for the specification of what he terms important learning goals (along a developmental progression). From a cognitive science perspective, the identification of critical transitions and their incorporation into the curriculum as goal statements is essential, since doing so allows for progressive conceptual development, from key conceptual foundations to the incremental construction of understanding. For example, children need to learn about units of quantification (Sophian, 2004) in ways that allow them to easily build on this knowledge as they meet new (key) ideas and as their concepts about these develop. While not previously made explicit as KDUs, Simon points to important examples of these in the literature. Amongst other work referenced by him in this respect is the work of Gelman and Gallistel (1978), Piaget (1952), and Steffe and Cobb (1988), all in the area of number. He sees their work as clearly identifying KDUs (cardinality, composite units and conservation of number) that are central to children's abilities to conceive of and work with number. Essentially what Sarama and Clements have done is to extract these from the literature and use them to build developmental progressions for their big ideas in mathematics.

Some efforts to identify key elements of domain-related content are to be found in the literature. In the United States the *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics* (NCTM, 2006) identified what they considered to be the important ideas and major themes which should receive special attention at particular points in time, across the domains of Number, Geometry and Measure. The aim of the focal points is to show teachers how they might build on important mathematical content and connections identified for each grade level (p. 3). The *Focal Points* approach 'focuses on a small number of significant mathematical targets for each grade level…[and] the most significant mathematical concepts and skills…' (p. 1). They are presented in narrative rather than list format, and describe the content emphases for different grade levels. Table 4.2 is an example of the kindergarten curriculum focal points.

Table 4.2. Kindergarten Curriculum Focal Points

Kindergarten Curriculum Focal Points	Connections to the Focal Points
Number and Operations: Representing, comparing, and ordering whole numbers and joining and separating sets. Children use numbers, including written numerals, to represent quantities and to solve quantitative problems, such as counting objects in a set, creating a set with a given number of objects, comparing and ordering sets or numerals by using both cardinal and ordinal meanings, and modeling simple joining and separating situations with objects. They choose, combine, and apply effective strategies for answering quantitative questions, including quickly recognising the number in a small set, counting and producing sets of given sizes, counting the number in combined sets, and counting backwards.	*Data Analysis*: Children sort objects and use one or more attributes to solve problems. For example, they might sort solids that roll easily from those that do not. Or they might collect data and use counting 'to answer such questions as, 'What is our favourite snack?' They re-sort objects by using new attributes (e.g., after sorting solids according to which ones roll, they might re-sort the solids according to which ones stack easily). *Geometry*: Children integrate their understandings of geometry, measurement, and number. For example, they understand, discuss, and create simple navigational directions (e.g., 'Walk forward 10 steps, turn right, and walk forward 5 steps').

Table 4.2. Kindergarten Curriculum Focal Points (*continued*)

Kindergarten Curriculum Focal Points	Connections to the Focal Points
Geometry: Describing shapes and spaces Children interpret the physical world with geometric ideas (e.g., shape, orientation, spatial relations) and describe it with corresponding vocabulary. They identify, name and describe a variety of shapes, such as squares, triangles, circles, rectangles, (regular) hexagons, and (isosceles) trapezoids presented in a variety of ways (e.g., with different sizes or orientations), as well as such three-dimensional shapes as spheres, cubes, and cylinders. They use basic shapes and spatial reasoning to model objects in their environment and to construct more complex shapes.	*Algebra*: Children identify, duplicate and extend simple number patterns and sequential and growing patterns (e.g., patterns made with shapes) as preparation for creating rules that describe relationships.
Measurement: Ordering objects by measureable attributes Children use measureable attributes, such as length or weight, to solve problems by comparing and ordering objects. They compare the lengths of two objects both directly (by comparing them with each other) and indirectly (by comparing both with a third object), and they order several objects according to length.	

Taken from *Curriculum Focal Points for Prekindergarten Through Grade 8 Mathematics: A Quest for Coherence*, by National Council of Teachers of Mathematics (NCTM) (2006), p. 12. Reston, Virginia: NCTM.

In Table 4.2. Kindergarten Curriculum Focal Points we see that the domains of Number, Geometry and Measure outline the key ideas within each domain. The key ideas are broken down into what appear to be critical transitions. These are framed as learning outcomes. It seems to us that the *Focal Points* approach provides a basis for structuring the curriculum at content level with the content-level descriptors providing a basis for identifying learning outcomes. We return to this topic in Report No. 18 (Chapter 3, Section: *Content Areas and Curriculum Presentation*).

Conclusion

The goals of a curriculum must be aligned with its underlying theory. A sociocultural stance implies that goals must be consistent with the view of learning as a socially and culturally embedded process which takes place in interaction with others. A curriculum which identifies goals and breaks them down into key mathematical ideas and critical transitions can help educators to move towards more focused teaching and assessment approaches.

The key messages presented in this chapter are as follows:

- Curriculum goals should reflect new emphases on ways to develop children's mathematical understandings, and to foster their identities as mathematicians. In the redeveloped curriculum both processes and content should be clearly articulated as goals.
- The approach whereby mathematical processes are foregrounded but content areas are also specified is consistent with a participatory approach to mathematics learning and development.
- General goals need to be broken down for planning, teaching and assessment purposes. Critical ideas derived in this way indicate the shifts in mathematical reasoning required for the development of mathematical concepts. An understanding of mathematical development enables teachers to provide support for children's progression towards curriculum goals.

These issues are addressed in *Chapter 5* and *Chapter 6*, and we return to them again in Report No. 18 (Chapter 3, Section: *Content Areas and Curriculum Presentation*). In *Chapter 5* we discuss different approaches to the specification of learning paths and teaching paths, designed to enable learners to progress towards the goals of the curriculum.

CHAPTER 5

The Development of Children's Mathematical Thinking

A Historical Perspective

The idea of stages of development in children's mathematical thinking and learning is most often associated with Piaget. His theory identified a sequence of what he considered to be invariant stages through which children's thinking progresses – from sensorimotor to pre-operational to concrete operational and finally formal operational. Each stage was characterised by a particular type of thinking applicable across many domains. But we now know that development is not equal across mathematical domains; for instance, children may conserve number before they can conserve mass or capacity (e.g., Ryan & Williams, 2007). Also, within domains, development is gradual rather than step-like (Casey, 2009). We know that the context, the materials, the task and especially the language used can make a difference to how children reason when faced with any mathematical task (e.g., Donaldson, 1984; NRC, 2005). Research also shows that contrary to Piaget's proposition, there is no clear progression from concrete to abstract thinking in children's development (e.g., NRC, 2009). Young children's thinking is both concrete and abstract (e.g., Ginsburg, 2009a).

One framework for mathematics learning and teaching that is receiving attention in countries as diverse as Japan, Korea, Australia, as well as in Europe and the United States is that of learning trajectories, also sometimes referred to as learning paths (e.g., Bobis et al., 2005; Daro et al., 2011; Griffin, 2004; Lewis & Tsuchida, 1998; Stigler & Thompson, 2012; van den Heuvel-Panhuizen, 2008). Interest in learning trajectories/learning paths is not confined to mathematics. They are also being developed in science and in literacy (e.g., Daro et al., 2011). The history of learning trajectories in mathematics education can be traced at least as far back as the work of Treffers (1987), whose perspective was that of the RME school (see below). The work of Simon (1995) was an important catalyst which resulted in intense interest in his (social constructivist) articulation of the concept of hypothetical learning trajectories (HLT). Work by American researchers Sarama and Clements is also included here since it currently features prominently in early childhood mathematics education, especially in the United States.

In this chapter we explore the progression from the Piagetian idea of stages of development to the idea of learning trajectories and learning paths. We see how, through the 1970s and 1980s, the idea of levels of mathematical thinking was developed as a concept of interest for Realistic Maths Education (RME) theorists. Then in the 1990s the concept of hypothetical learning trajectories (HLTs) was advanced by Simon. He saw HLTs as key elements in mathematics teaching cycles. More recently, in the United States, Sarama and Clements have developed their learning trajectories for learning and teaching early mathematics (e.g. 2009). Each of these developments is of interest in the context of the current review, and potentially informative in relation to issues of curriculum, assessment, equity and teacher education.

From Stages of Development to Levels of Sophistication in Thinking

In recent decades cognitive scientists have focused on knowledge construction and on the thinking that children use to solve problems. This concerns children's internal cognitive structures and processes and researchers' interpretations and understandings of what is happening in relation to the child's thinking (Cobb, 2007). Piaget's theory has been adapted to gain insights into children's mathematical thinking and how that thinking changes and develops over time. Interests are focused on how change occurs, most significantly qualitative changes in children's mathematical reasoning (e.g., Casey, 2009). Both constructions of meaning for specific mathematics topics and the characterisation of children's developing conceptualisation and reasoning in terms of different levels of sophistication in thinking are important emphases (Battista, 2004, p. 186).

Developing Children's Mathematical Thinking: Three Approaches

An emphasis on helping learners to move through increasingly sophisticated levels of mathematical reasoning and understanding is now seen as a key focus for mathematics education from a cognitive science point of view (e.g., NRC, 2009). Gravemeijer (2004) argues that a pedagogy which supports this is generally well-articulated, i.e., it is 'elaborated in terms of classroom culture, social norms, mathematical discourse, mathematical community, and a stress on inquiry and problematizing' (p. 106). However, he argues that it is necessary to draw attention to the curriculum counterpart of this innovative pedagogy. He points out that in the 1960s and 70s curriculum design took as its starting point the knowledge and expertise of experts in order to construct learning hierarchies. The problem with that approach was that it did not take into account the perspective and personal input of the learner. Proposed revisions to the mathematics curriculum will need to consider how to ensure that this issue is addressed, particularly in guidance on pedagogy.

Below, we present three different approaches to helping teachers in the task of developing children's mathematical thinking in the way described above. What they have in common is the fact that each subscribes to the idea of learning trajectories or learning paths. Where they diverge is in the roles they see these playing in the teaching/learning process.

The First Approach: Working with Children's Thinking and Understanding (RME)

The notion of levels of thinking was first advanced by Freudenthal who drew, in particular, on the work of Pierre and Dina van Hiele. They were his students, and they had developed a model of geometric thinking at the University of Utrecht, Netherlands in 1957 (Crowley, 1987). The basis of this model is that thinking develops from an initial visual level through increasingly sophisticated levels, that is, analysis, abstraction, deduction and rigour.

Freudenthal (1971) expanded on this model in his theory on the learning of mathematics:

> *The van Hiele levels of the learning process are often characterised by a logical feature: the activity on one level is subjected to analysis in the next, the operational matter on one level becomes a subject matter on the next level.* (p. 417)

This means that mathematical activities that have been carried out in an informal way initially later become more formal as a result of reflection (this is an aspect of mathematization as described by RME theorists e.g., van den Heuvel-Panhuizen, 2003). Early mathematics is constituent of and not separate from formal mathematics, implying that RME ideas about levels of thinking and their implications for pedagogy are elaborations of children's earlier understandings.

Key Features

The RME approach entails directing teachers' attention to children's understandings of mathematics and engaging children with rich problem contexts. Instruction evolves to suit the learners. When first introduced in the 1970s, this was a novel way to approach teaching. A feature of the approach is that children work with realistic problems. These allow them to imagine. The problems can include contexts from real-world situations, but also problems from the fantasy world of fairy tales or from the formal world of mathematics (van den Heuvel-Panhuizen, 2003). A second essential feature is the use of models developed by the children as a basis for teaching and learning. These have a specific role in that they provide the context in which children can be supported in the activity of mathematizing, i.e., 'the analysing of real world problems in a mathematical way' (Treffers & Beishuizen, 1999, p. 32). A third feature is that different levels of understanding can be distinguished and as children pass through these levels, models can have an important role in level-raising: they are seen as bridges between informal understanding and the abstraction of formal ideas. A model can, for instance, include materials, visual sketches or symbols. Models share two important characteristics: they have to be rooted in realistic contexts and they must be flexible

and applicable on a more general level. Models can be models of a situation initially, but then they must be capable of becoming models for organising new problems and reasoning about these in a mathematical way (van den Heuvel-Panhuizen, 2003). The models are formulated by children themselves in the course of their engagement with the problem and they gradually gain a better understanding of a rich, meaningful problem situation by describing and analysing it with more and more advanced means. By going through a series of modeling cycles, they finally develop an effective model with which they can also take on other (similar) complex problem situations (p. 29). See Report No. 18, Chapter 2, Section: *Emphasis on Mathematical Modeling*.

The Teacher's Task

The RME position is that levels of thinking or understanding can be specified in a general sense and it is the teacher's task to explore children's understandings at the different levels and use these to progress learning.

While general hypothetical learning trajectories are used as the basis of the teacher's work these are seen as initial starting points which are subject to constant revision by the teacher as a learning trajectory specific to his or her particular classroom emerges. From this perspective, teachers learn to use learning-teaching trajectories that fit their particular situation (Gravemeijer, 2004). The trajectory provides an overview of levels of understanding in a domain. It should not be seen in a linear way: there can be variations in the steps. The trajectory sets out important signposts, and allows teachers to discern the differences in children's understandings. This approach is very much about developing the teacher's abilities to make decisions about how best to help children with 'intermediate attainment targets', on the way to achieving general goals. These are seen as the crucial steps or 'landmarks towards which the learning can be oriented' (van den Heuvel-Panhuizen, 2008, p. 9).

Initial work in their project of developing learning-teaching trajectories has focused on the domain of number since it is seen as an area of concern for teachers and a good place to begin. This work is shown in Table 5.1 below. Some work has also been done on Geometry and Measures (van den Heuvel-Panhuizen & Buys, 2008). See also Report No. 18, Chapter 3, Sections: *Measurement*; *Geometry and Spatial Thinking*.

Table 5.1: A Learning-Teaching Trajectory for Number

Elements	Levels
Emergent numeracy (preschool)	
▪ Recognising 'two-ness', 'three-ness', and 'many-ness' as a property of a group of objects ▪ Learning to recall the number sequence ▪ Imitating resultative counting ▪ Symbolising by using fingers	
Growing number sense (K1 and K2)	
▪ Learning to count	Children know the counting sequence, at least up to 10.
▪ Learning to count and calculate	Within what are for them meaningful context situations, children are able to count to at least 10, arrange numbers in the correct order, make reasonable estimates, and compare quantities being more, less or equal (level 1).
▪ Context bound counting and calculating	Children can order, compare, estimate and count up to 10 objects. They are also able to select a suitable strategy for simple addition or subtraction situations in such things as concealment games for up to 10 objects (level 2).
▪ Towards pure counting-and calculating via symbolisation	Children can represent physical numbers up to 10 on their fingers and with lines and dots, and are able to use these skills for 'adding up' and 'taking away'.
Calculations up to 20 (G1 and G2)	
▪ Calculations by counting, supported where necessary by counting materials ▪ Non-counting based calculating by structuring with the help of suitable models	The children can recite the number sequence up to 20 and can count up and down from any number in this domain. They can also put numbers up to 20 into context by giving them a real world meaning, can structure then by doubling and using groups of five and 10, and place them on an empty number line from 0 to 20.
▪ Formal calculation using numbers as mental objects for smart and flexible calculation without the need for structured materials	The children should be able to add and subtract quickly, in the number area up to 20 by structuring the numbers and, in time, they should be able to perform formal calculations with the help of remembered number properties. They should also be able to use this skill in elementary context situations and be able to both understand and use some conventional mathematical notation.

Adapted from van den Heuvel-Panhuizen, 2008

By way of defining a learning-teaching trajectory, van den Heuvel-Panhuizen (2008, p. 13) states that there are three interwoven meanings:

- a learning trajectory that gives a general overview of the learning process of the students
- a teaching trajectory, consisting of didactical indications that describe how the teaching can most effectively link up with and stimulate the learning process
- a subject matter outline, indicating which of the core elements of the mathematics curriculum should be taught.

van den Heuvel-Panhuizen (2008) describes how the learning-teaching trajectory, or TAL[8], with intermediate targets for calculation with whole numbers in primary schools builds on children's earlier numerical experiences. They present TALs for number for the youngest children at three levels. They call the first level the level of 'Emergent numeracy' (preschool years), the second the level of 'Growing number sense '(kindergarten 1 and 2), and the third 'Calculations up to 20' (grades 1 and 2). Further discussion of the relevance of this work as it applies to Number, Geometry and Measures is presented in Report No. 18 (Chapter 3, Section: *Content Areas*).

The intention is to extend this work into secondary education. The learning-teaching trajectory is seen as part of 'the longitudinal perspective' (p. 11) that all teachers need to hold. It is seen to go beyond a textbook and beyond tests, but to focus on the attainment targets and a general indication of teaching activities that can contribute to achieving these. In particular, domain-specific 'levels' of understandings are seen as potentially useful for specifying communal trajectories, i.e., ones that apply to particular school years or grades. They are also seen as useful in relation to level-raising, i.e., moving children towards the final core goals of mathematics education at primary level. In addition, it is suggested that TALs provide a means whereby teachers can monitor children's development.

The Second Approach: Teacher-Generated Hypothetical Learning Trajectories (Simon)

Working with the tension created by the need to attend to predetermined goals for children's learning whilst at the same time being responsive to children's thinking, Simon (1995) developed a theoretical model of teacher decision-making with respect to mathematics tasks. Simon-proposed HLTs comprise the learning goal, the learning activities and a description of the thinking and learning that students might engage in.

8 In Dutch, learning-teaching trajectories are referred to as TALs. TAL stands for Tussendoelen Annex Leerlijnen, translated into English as *intermediate attainment targets in learning-teaching trajectories* (Fox, 2005/2006).

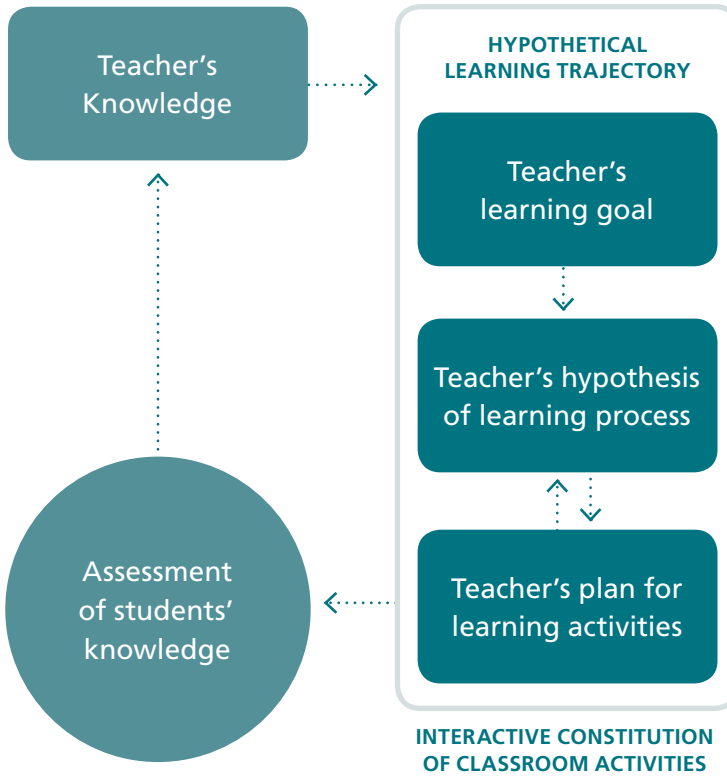


Figure 5.1. Adapted from Reconstructing Mathematics Pedagogy From a Constructivist Perspective by M. Simon, 1995. *Journal for Research in Mathematics Education*, 26(2), p. 136.

He emphasised the unpredictable nature of teaching mathematics, and the need for continuous modifications to the teaching plan. Simon (1995) describes a dynamic cycle wherein:

> *as students begin to engage in the planned activities, the teacher communicates and observes the students, which leads the teacher to new understandings of the students' conceptions. The learning environment evolves as a result of interaction amongst the teacher and students as they engage in the mathematical content…it is what the students make of the task and their experience with it that determines the potential for learning.* (p. 133)

Simon's explication of the term hypothetical learning trajectory emphasises the teacher's prediction as to the path by which learning might proceed. It emphasises its hypothetical nature – the actual learning trajectory is not known in advance. For Simon, learning trajectories are essentially a teaching construct. This is similar to the RME perspective. While Simon's (1995) original articulation of the HLT did not specify teaching activities, he did later address the issue of tasks (Simon & Tzur, 2004). His concern then was that tasks would not be left to intuition or trial and error but would be deliberately constructed to promote the learning process.

The teacher, in designing a learning trajectory, must consider both the tasks to be used and the learning goals. With respect to the conceptual learning goals, Simon (2006) proposes the identification of key developmental understandings (KDUs):

> *significant landmarks in students' mathematical development…understandings that account for differences between those learners who show evidence of more sophisticated conceptions from those who exhibit less sophisticated conceptions.* (p. 370)

The significance of Simon's approach is the influence it had on subsequent work on ways of developing children's mathematical thinking. One example of this is the body of work on learning trajectories developed by Sarama and Clements.

The Third Approach: Pre-Specified Developmental Progressions as a Basis for Learning Trajectories (Sarama and Clements)

In the United States, Sarama and Clements synthesised relevant research from cognitive science on the learning of mathematics from birth to age 8. From this synthesis they developed their learning trajectories (e.g., 2009). Table 5.2 shows Sarama and Clements' learning trajectory for volume measurement. We draw the reader's attention to the level of detail presented in these developmental progressions and the accompanying hypothesised mental actions that appear in the third column. Note also the age column on the left. The authors state that these are age-indicators based on research and are provided only as a general guide.

Table 5.2. A Developmental Progression for Volume Measurement

Age (Years)	Development Progression	Actions on Objects
0–3	*Volume/Capacity:* **Volume Quantity Recognizer** Identifies capacity or volume as attribute. ▪ Says, 'This box holds a lot of blocks!'	Perceives space and objects within the space.
4	**Capacity Direct Comparer** Can compare two containers. ▪ Pours one container into another to see which holds more.	Using perceptual objects, internal bootstrap competencies to compare linear extent (see the length trajectory for 'Direct Comparer') or recognize 'overflow' as indicating the container 'poured from' contains more than that 'poured into.'
5	**Capacity Indirect Comparer** Can compare two containers using a third container and transitive reasoning. ▪ Pours one container into two others, concluding that one holds less because it overflows, and the other is not fully filled.	A mental image of a particular amount of material ('stuff') can be built, maintained, and manipulated. With the immediate perceptual support of the containers and material, such images can be compared. For some, explicit transitive reasoning may be applied to the images or their symbolic representations (i.e., object names).
6	*Volume/Spatial Structuring:* **Primitive 3-D Array Counter** Partial understanding of cubes as filling a space. ▪ Initially, may count the *faces* of a cube building, possibly double-counting cubes at the corners and usually not counting internal cubes. ▪ Eventually counts one cube at a time in carefully structured and guided contexts, such as packing a small box with cubes.	With perceptual support, can visualize that 3-D space can be filled with objects (e.g., cubes). With strong guidance and perceptual support from pre-structured materials, can direct the filling of that space and recognize that filling as complete, but often only intuitively. Implicit visual patterning and constraints of physical materials guides placement of cubes.

Table 5.2. A Developmental Progression for Volume Measurement (*continued*)

Age (Years)	Development Progression	Actions on Objects
7	**Capacity Relater and Repeater** Uses simple units to fill containers, with accurate counting. ▪ Fills a container by repeatedly filling a unit and counting how many. ▪ With teaching, understands that fewer larger than smaller objects of units will be needed to fill a given container.	See the learning trajectory level, Length Unit Relater and Repeater.
7	*Volume/Spatial Structuring:* **Partial 3-D Structurer** Understands cubes as filling a space but does not use layers or multiplicative thinking. Moves to more accurate counting strategies e.g.: ▪ Counts unsystematically, but attempts to account for internal cubes. ▪ Counts systemically, trying to account for outside and inside cubes. ▪ Counts the numbers of cubes in one row or column of a 3-D structure and using skip counting to get the total.	Builds, maintains, and manipulates *mental* images of composite shapes, structuring them as composites of individual shapes and as a single entity – a row (a unit of units), then a layer (a 'column of rows' or unit of unit of units). Applies this composite unit repeatedly, but not necessarily exhaustively, as its application remains guided by intuition.
8	*Area/Spatial Structuring:* **3-D Row and Column Structurer** ▪ Counts or computes (row by column) the number of cubes in one row, and then uses addition or skip counting to determine the total. ▪ Computes (row times column) the number of cubes in one row, and then multiplies by the number of layers to determine the total.	Builds, maintains, and manipulates *mental* images of composite shapes, structuring them as composites of individual shapes and as a single entity – a layer (a unit of unit of units) of *congruent* cubes. Applies this composite unit repeatedly *and exhaustively* to fill the 3-D array – coordinating this movement in 1–1 correspondence with the elements of the orthogonal column. If in a measurement context, applies the concept that the length of a line specifies the number of unit lengths that will fit along that line. May apply a skip counting scheme to determine the volume.

Taken from *Early Childhood Mathematics Education Research: Learning Trajectories for Young Children* by J. Sarama and D. Clements, 2009, pp. 306–607. New York, NY: Routledge.

The key difference between their work and that of either the RME school, or Simon, is the emphasis they place on developmental progressions. These are learning paths 'through which children move through levels of thinking' (2009, p. 17).

Clements and Sarama (2004) set out to emphasise both learning processes and teaching processes together:

> *We conceptualise learning trajectories as descriptions of children's thinking and learning in a specific mathematical domain and a related conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesised to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain.* (p. 83)

While initially they used the term hypothetical learning trajectories, more recently they tend to use the term learning trajectories while still maintaining that their trajectories are hypothetical. As discussed earlier, Sarama and Clements see their goals as the twelve big ideas they identify for early mathematics (see Chapter 4, *Table 4.1. Specifying Goals: Different Approaches*). The research-based developmental progressions or learning paths identify the levels of thinking that children progress through as they work towards the goal. These levels of thinking are at the core of the trajectory. The instructional tasks or teaching paths consist of 'sets of instructional tasks, matched to each of the levels of thinking in the progressions' (Clements & Sarama, 2009a, p. 2). Also, as noted above, age estimates are also provided as a general guide to when children might develop certain understandings.

The levels of thinking, as characterised by Clements and Sarama (2009b), are understood to be domain-specific:

> *Children are identified to be at a level when most of their behaviours reflect the thinking-ideas and skills of that level…Levels are not absolute stages. They are benchmarks of complex growth that represent distinct ways of thinking…sequences of different patterns of thinking and reasoning. Children are continually learning, within levels and moving between them…Children may also learn deeply and jump ahead several 'levels' in some cases.* (p. 5)

Comparing the Three Approaches

Definitions and Characteristics

The learning trajectory concept is interpreted, re-presented and applied in a range of different ways. Table 5.3 presents the definitions of learning trajectories for each of the three approaches described above. We see these definitions as indicative of some of the subtle nuances and differences inherent in each of the approaches.

Table 5.3. Three Approaches to Defining Learning Trajectories

Realistic Maths Education Trajectories	Simon's Hypothetical Learning Trajectories	Sarama and Clements' Trajectories
A learning-teaching trajectory has three interwoven meanings: ▪ a learning trajectory that gives a general overview of the learning process of the students ▪ a teaching trajectory, consisting of didactical indications that describe how the teaching can most effectively link up with and stimulate the learning process ▪ a subject matter outline, indicating which of the core elements of the mathematics curriculum should be taught.	A hypothetical learning trajectory is composed of the learning goal, the learning activities and a description of the thinking and learning that students might engage in.	Learning trajectories are descriptions of children's thinking and learning in a specific mathematical domain and a related conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesised to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain (Clements & Sarama, 2004, p. 83).

Learning-teaching paths from the RME perspective have much in common with sociocultural/situative perspectives. For example, establishing an appropriate classroom culture for successful learning in mathematics is emphasised. Discussion is the context within which the teacher focuses on what Gravemeijer (2004) refers to as 'the inventions of the students' (p. 126). The approach used by Simon and by Sarama and Clements both take a cognitive science approach to promoting children's mathematical understanding. Simon's theoretical notion of HLTs was important in that it moved away from any notion of learning progressing in a linear way. It recognised that not all children follow the same path towards understanding. It sought to apply constructivist theory to teaching approaches. Similarly, Fosnot and Dolk (2001) describe what they call 'a landscape of learning' and how children traverse this as they engage in mathematics in the classroom – 'They go off in many directions as they explore, struggle to understand, and make sense of their world mathematically' (p. 18).

Fosnot and Dolk describe how children's learning paths twist and turn, cross each other and often use an indirect route to get a particular landmark. They illustrate this in Figure 5.2 below.

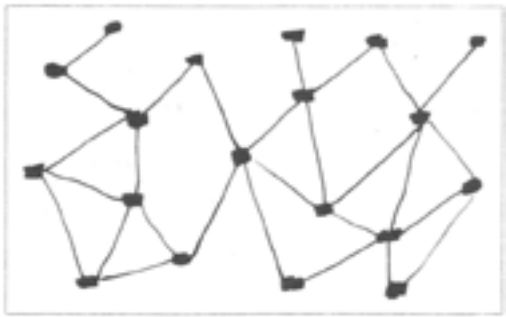


Figure 5.2: Reprinted with permission from Young Mathematicians at Work: Constructing Number Sense, Addition and Subtraction by C. T. Fosnot & M. Dolk, 2001, p. 18. Portsmouth, NH: Heinemann. All rights reserved.

In contrast, because the detailed learning trajectories developed by Sarama and Clements are presented linearly, educators may incorrectly infer that mathematical development is linear.

It seems to us that one of the confounding issues for readers in dealing with the dense and complex literature surrounding the various learning trajectories and learning paths is that, while they are referred to as learning trajectories, their central purpose is generally as a pedagogical tool. Simon and Tzur (2004) clearly state that a HLT is a vehicle for planning the learning of concepts, while Clements and Sarama (2004) consider Simon's HLTs as a way of describing the pedagogical thinking involved in teaching mathematics for understanding.

The different ways in which the trajectories model has been developed is perhaps also a response to the context in which individual theorists see the trajectories being used. For instance, in the US there is a great deal of concern to accelerate both the professional development of teachers of children aged 3–6 years, and to specify standards for early childhood mathematics education. The learning trajectory approach is seen as a way to meet both concerns (e.g., NRC, 2009). On the other hand, RME perspectives appear to focus more on working in a contingent way with children's ideas. Theorists from this perspective generally see learning trajectories, or TALs, as a tool to be used in contexts characterised by teacher judgement, and where teaching is characterised by an emergent, creative and adaptive pedagogy focused on real problems located in children's own experiences. In Japan, for over a decade, learning trajectories have provided the basis for lesson study, i.e., detailed planning of research lessons by teachers. These lessons are taught, reflected on, analysed and redesigned by teachers and in this way instruction improves (Lewis & Tsuchida, 1998). See Report No. 18, Chapter 6, Section: *Frameworks for Thinking about Pedagogy; Using Tools for Teacher Preparation*.

Recognising Diverse Routes in Learning

From a sociocultural point of view, there are many possible routes that children may take to reach a common goal. Socioculturists emphasise the role that experiences and contexts play in determining what children learn, but also the role of context in determining what learning children might display

during observations and assessments focused on ascertaining the extent of their understanding. Sarama and Clements (2009) describe how, in constructing their developmental progressions, they used the available research but, where none was available, they used judgement and best guesses to suggest a hypothetical path. Learning trajectories then can be regarded as 'invented cultural artefacts' that have been constructed in order to 'help students get from point A to point B' (Stigler & Thompson, 2012, p. 192). Taking a similar stance, Wager and Carpenter (2012) remind us that learning trajectories are cognitive constructs based on certain assumptions about the cognitive nature of knowledge '… they do not fully account for the situated nature of children's learning…they should be used in a way that considers and connects to children's experiences' (p. 198). Another observation suggests that learning trajectories based on tightly specified developmental progressions appear to have lost Simon's original focus on children's learning as it might unfold in interactions with the teacher and the accompanying decision-making that the teacher might engage in (Empson, 2011). In contrast, the TALs are much less detailed and thus explicitly suggest that development can follow different paths.

Our reflections suggest that any presentation of learning trajectories to educators would need to be couched in terms of their potential as reference tools and not as roadmaps.

Recognising Developmental Variation

The different ways in which the learning trajectories have been developed are to some extent a consequence of the perspectives of the theorists and the extent to which they subscribe to different social and/or cognitive perspectives. The idea of universal development is deeply ingrained in cognitive science and the idea that many children may do things differently during the course of their mathematical learning and development is a relatively new expectation influenced by sociocultural perspectives. The following joint statement from the US *National Association for the Education of Young Children/National Council for Teachers of Mathematics* (2002/2010) is helpful in stating a balanced position:

> *the research base for sketching a picture of children's mathematical development varies considerably from one area of mathematics to another. Outlining a learning path does not mean that we can predict with confidence where a child of a given age will be in that sequence. Developmental variation is the norm, not the exception. However children do tend to follow similar sequences, or learning paths, as they develop.* (p. 19)

We have seen, however, that some frameworks use age-related steps or indicators in order to present learning progressions. We see inherent dangers with this approach. The key point we wish to emphasise here is that the linking of stages of development with age-levels is problematic. Different children develop at different rates and their learning is strongly influenced by culture and experience. From our perspective, it is more theoretically coherent to conceive of development and learning as proceeding along a path which has significant markers. The learning trajectory or path for an individual child cannot be known in advance. In other words, any proposed trajectory is a

hypothetical one. Hypothetical learning trajectories are not the same thing as preferred teaching trajectories or paths. We have seen that, according to the original construct, hypothetical learning trajectories are a teaching construct which are speculative as regards how a particular child may develop. On the other hand, preferred teaching trajectories or paths could be considered a useful framework of reference for planning on the part of teachers.

Curriculum Development and the Role of Learning Trajectories

Steffe (2004) raises the question, 'Whose job is it to design learning trajectories?'. First and foremost, responses to this question are reflective of a view of knowledge, of learning and of teaching. They also reflect understandings about teachers and teaching, and of autonomy and agency in relation to the profession. They relate to issues of teacher preparation and preparedness in working with learning trajectories, in conceptualising children's mathematical learning, in planning effectively, and in establishing an appropriate classroom culture for successful learning in mathematics.

Supporting Teachers in Planning

The mathematics curriculum is concerned with emphasising tasks that enable children to work in different ways, to organise and interpret tasks in ways that make sense to them while making use of different mathematical strategies. This necessitates the design of HLTs. Designing these is not an easy task. The teacher must understand children's mathematical conceptions and engage in conceptual analysis (e.g., Simon & Tzur, 2004, Clements & Sarama, 2004). In recognition of the need to support teachers in this regard, Gravemeijer (2004) proposed that they be offered 'a framework for reference and a set of exemplary instructional activities that can be used as a source of inspiration' (p. 107). Ready-made instructional sequences are rejected because the teacher will continually have to adapt to the actual thinking and learning of his or her students. The emphasis is on the local nature of the planning. The trajectories are developed in response to the children's ideas and follow the cyclical process outlined by Simon (1995) and described above. Clements and Sarama (e.g., 2009b) appear to take a somewhat different approach, one where much of the decision-making is done by mathematics educators and presented to teachers in the form of detailed specifications of teaching paths. It seems to us that mathematics education theorists, in dealing with this quandary of teachers' understanding and their generation of learning trajectories, have taken diverging approaches. The issue is really about the detailed specification of what Clements and Sarama (2004) describe as 'natural developmental progressions' (p. 83). While these authors and others coming from a mainly cognitive science perspective see such specification as unproblematic, theorists coming from a sociocultural or similar perspective (for example, RME) are likely to temper such a position in favour of an approach which emphasises more explicitly the hypothetical nature of learning paths.

While conceding that detailed developmental sequences are most likely over-simplified descriptions of development, we see them as having a role in terms of assessment. They can provide a theoretical framework for guiding teacher judgements (e.g., Ginsburg, 2009b). Their strength here lies in the fine-grained analysis of learning that they provide. They can serve as reference points as to where children are along the way to meeting the goals of the curriculum (e.g., Daro et al., 2011). They can provide a structure within which teachers can identify and address difficulties that arise for children. HLTs are seen as particularly useful for teaching concepts whose learning is problematic generally or for particular students (Simon & Tsur, 2004). One identifiable gap in the literature is the use of these trajectories for identifying and addressing the needs of high-achieving learners.

An example of the use of learning trajectories to develop teachers' work in assessing young children aged 5–8 years is provided by the Victorian Early Numeracy Research Project (ENRP). The three year project focused on developing teachers' understanding of mathematics in the early years, evaluating the effect of professional development programmes, and describing effective practice in mathematics in the early years of schooling (Clarke, 2001; Bobis et al., 2005). Central to the ENRP was the development of a framework of 'growth points' in young children's understandings of mathematics in different domains. Growth points were considered by the ENRP team as 'key stepping stones' along paths to mathematical understanding (Clarke, 2001). It was not considered that children would necessarily pass through each growth point in succession or that the growth points were discrete. Furthermore, the framework gave teachers a tool for assessing children's understandings and building on children's current skills and concepts. One of its purposes was to provide a basis for task construction for assessment via interview. In developing this framework the researchers drew on the work on learning trajectories. Assessment tasks were created to match the framework.

The issue of learning trajectories and assessment is also discussed in Chapter 6 in the context of assessing and planning for progression (Section: *Supporting Children's Progression with Formative Assessment*).

Supporting Learning for Pre-Service Teachers

We also see an important role for well-structured developmental progressions of concepts in the education of teachers, particularly at pre-service level. Detailed knowledge of these can provide pre-service teachers with frameworks related to general mathematics development.

Conclusion

In considering the potential of the range of work on learning trajectories, we find it useful to consider Fosnot and Dolk's (2001) perspective on the issues which are implicated in teachers' understandings of children's learning and of how to plan for that learning. Fosnot and Dolk argue that

> *Strategies, big ideas and models are all involved – they all need to be developed as they affect one another. They are the steps, the shifts and the mental maps in the journey. They are the components in a "landscape of the learning".* (p. 12)

The research indicates that teachers' understanding of developmental progressions is one aspect in helping them to develop hypothetical learning paths for use in their classrooms. They sit alongside their knowledge of the big ideas or key goals (see *Chapter 4*). They support teachers' understandings of children's emerging models (see this chapter). Research also suggests that teachers need a great deal of support in moving from a linear model of learning to one in which children engage as members of dialogic communities in tasks that are truly problematic (see Fosnot & Dolk, 2001). All of this has implications for teacher education, an issue that is discussed in Report No. 18 (*Chapter 6*).

The key messages arising from this chapter are as follows:

- Learning trajectories describe learning paths in the various domains of mathematics. These are based on developmental progressions which have been constructed for a number of big ideas in mathematics. They indicate a general sequence that might apply to development.
- There are different approaches to the explication of learning paths. For example, linear/nonlinear presentation, level of detail specified, mapping of paths to age/grade, and role of teaching. Different presentations reflect different theoretical perspectives.
- An approach to the specification of learning paths that is consistent with sociocultural perspectives is one which recognises the paths as

    i. provisional, as many children develop concepts along different paths and there can never be certainty about the exact learning paths that individual children will follow as they develop concepts.

    ii. not linked to age, since this suggests a normative view of mathematics learning.

    iii. emerging from engagement in mathematical-rich activity.

Curriculum design must take into account the children's reasoning in and contribution to the learning-teaching situation.

CHAPTER 6

Assessing and Planning for Progression

This chapter looks at the assessment of mathematics and ways in which assessment data can be used in planning for progression in mathematics learning in preschool and primary school settings attended by young children. First, the chapter examines formative assessment in terms of conceptual underpinnings and key methods. The focus then shifts to diagnostic and summative assessment as the use of screening/diagnostic and standardised tests is considered. The chapter concludes with a consideration of the use of assessment data for planning and progression in a range of contexts, including immersion settings, and settings involving children with special needs.

The formative assessment methods discussed include observation, tasks, interviews, conversations and pedagogical documentation. The methods are inclusive of all children. Each of these provides scope for examining the embedded nature of children's mathematical learning, changes in their understandings, what children can do when supported by others, their potential capabilities and strengths, and their participation in activities and tasks. They also provide scope for assessing children's dispositions and identities as mathematics learners. These aspects are key foci of assessment from a sociocultural perspective (e.g., Fleer & Richardson, 2009). The discussion on screening/diagnostic tests urges care in using such methods, and highlights a need to draw on multiple sources of information when assessing children in various at-risk groups. Caution is advised in relation to the use of standardised tests with children in the 3–8 years age range.

Assessing Mathematics Learning in Early Childhood

Aistear (NCCA, 2009b) defines assessment as

> *the on-going process of collecting, documenting, reflecting on and using information to develop rich portraits of children and learners in order to support and enhance their future learning.* (p. 72)

Assessment in the Primary School Curriculum: Guidelines for Schools (NCCA, 2007) offers a similar vision of assessment.

In order to support children's learning, it is essential that teachers are familiar with each child's mathematical understandings and learning. Educators acquire this understanding through formative assessment of children's mathematical learning since this approach serves to best represent the

complexity and depth of children's learning (e.g., Carr 2001; Carr & Lee, 2012; Drummond 2012; Perry, Dockett & Harley, 2007). Increasingly, assessment is seen as a collaborative process between children and adults, and one in which teachers support and scaffold children's work. This view of assessment is predicated on a view of pedagogy that has relationships at its core (e.g., Fiore, 2012).

Formative Assessment

Conceptual Frameworks

Eliciting children's mathematical thinking is critical to understanding, monitoring and guiding their mathematical learning. Research-based conceptual frameworks which describe mathematical thinking in terms of levels of sophistication (i.e., learning paths as described in *Chapter 4*), provide the basis against which educators can then interpret children's reasoning. This process of locating a child's thinking on what Battista (2004, p. 202) refers to as 'a detailed map of the cognitive terrain required to construct understanding of a topic' is referred to as cognitive-based assessment, and it is increasingly seen as an effective tool for planning learning opportunities and for guiding children in their construction of mathematical meaning. We also know that dispositional learning is a crucial aspect of early learning and this too must be monitored and fostered. Carr and Lee (2012) illustrate the centrality of dispositions when they state that 'Dispositions act as an affective and cultural filter for the development of increasingly complex knowledge and skills' (p. 15). In other words, children's dispositions towards mathematics and towards engaging in mathematical ways of thinking and knowing are influenced by how they feel towards these activities. Knowledge and dispositions develop hand in hand – they are interdependent. Formative assessment also plays a key role in the construction of a learner identity (e.g., Bruner, 1996; Carr & Lee, 2012). Identity develops as children interact with mathematical knowledge, skills and ideas in the home and in education settings – it is socioculturally constructed. This implies that children's identities as mathematics learners are formed during early childhood. Learning and a sense of identity cannot be separated; some consider them one and the same thing (Lave & Wenger, 1991). How teachers and parents recognise and respond to children's numeracy practices shape children's identities (e.g., Anderson & Gold, 2006). Educators can greatly influence the development of children's identities as mathematicians by the way in which they frame children's activity. For instance, children will bring a rich store of mathematical achievement with them to school. This needs to be recognised and harnessed. Carr and Lee (2012) remind us of the opportunities that educational settings provide not just for the construction of identity but also critically for the editing of learner identities. In other words, teachers can influence, in a positive way, children's perceptions of themselves as mathematics learners. One way to do this is for teachers to collect and study mathematics-related vignettes of children's social activities at home and in the education setting – and then to reflect on the meanings of these. This could be especially effective for supporting children during transitions and during the early months in a new setting, particularly when discussed with children and parents.

In the section that follows we focus on what research tells us about how caregivers and teachers can most effectively carry out assessments of learning in order to gather data on children's achievement and their developing dispositions and identities as mathematicians.

Methods

In line with good early childhood practice internationally, both *Aistear* (NCCA, 2009b) and *Assessment in the Primary School Curriculum: Guidelines for Schools* (NCCA, 2007) identify a range of appropriate methods of formative assessment including observations, conversations, tasks, tests and self-assessment. Educators can assemble portfolios of children's learning and they can work with children and parents to compile pedagogical documentation as evidence of children's mathematics learning. Effective assessment is closely related to teachers' knowledge and their recognition of what constitutes significant learning, some of which could be informed by their knowledge of general learning paths in the major mathematical domains. A number of methods can be used, often together, to build a rich picture of children's mathematical learning over time. The ability to recognise the mathematics in children's everyday activities and to extend the potential learning arising from these is critical.

Observations

Observations can provide educators with the data to write rich narrative assessments of children's mathematical learning. These assessments can focus on different aspects of children's mathematical development. Contextual information can be included in the emerging picture of children's development. Depending on the circumstances, questioning or follow-up tasks can be used in order to check children's levels of mathematical understanding demonstrated or assumed. In engaging in these processes, educators draw on their deep knowledge of what mathematics is and how it develops in early childhood (e.g., Ginsburg & Ertle, 2008).

Arising from observations, 'learning stories' (Carr, 2001) can be constructed by the educator or co-constructed by the educator and child/children, with contributions from family and other significant adults. These are narrative accounts of learning and development and they take a holistic approach to assessment. They are often supplemented with photographs.

Carr (2000) describes learning stories as 'structured observations, often quite short, that take a 'narrative' or 'story approach' (p. 32). They keep the assessment anchored in the situation or action. Learning stories are rich and deep accounts of selected events as they are observed through specific lenses, for example the themes or goals of the curriculum. These assessments are learner-centred as opposed to content-centred. They do not fragment children's learning and they pay attention to the positive, rather than focusing on need and deficit (e.g., Dunphy, 2008).

When initially developed, learning stories focused mainly on dispositional learning (e.g., Carr, 2001). However, recent developments of the method in early childhood classrooms in schools in New

Zealand have seen teachers focus on both knowledge and disposition. This involves the teacher noting both the mathematics and the learning disposition evident in the analysis (Carr & Lee, 2012). From their experiences in working with preschool- and school-based educators, these authors conclude that

> *Learning Stories can capture the intermingling of expertise and disposition, the connections with the local environment that provide cues for further planning, the positioning of the assessment inside a learning journey, and the interdependence of the social, cognitive and affective dimensions of learning experiences. At the same time, Learning Stories enable children and students to develop capacities for self-assessment and for reflecting on their learning.* (p. 131)

In addition, Carr and Lee argue that learning stories meet four challenges associated with formative assessment: the challenge of engaging children in co-authoring the curriculum and assessment and exercising agency in relation to aspects of their learning; encouraging reciprocal relationships with families; recognising learning journeys and continuities in learning over time; and appropriating a repertoire of practices where the learning is distributed over a number of languages and other modes of meaning making. Even the youngest children are now becoming everyday users of technology in the home and in early education settings (e.g., Plowman, Stephen & McPake, 2010). Learning mathematics with technology, and using technology to express mathematics understanding and thinking are increasingly important avenues of learning and expression for young children (see Report No. 18, Chapter 2, Section: *Digital Tools*). Arising from their work with teachers, Carr and Lee (2012) observe that

> *Learning Stories have now participated in the new digital technologies in three ways: transforming the ways in which Learning Stories can be constructed, tracing children's Information Communications Technology (ICT) learning journeys, and emphasising the value of image-based ways of thinking.* (pp. 112–113)

From the assessment perspective, this expands the ways in which children's learning can be identified and documented. It provides a multi-modal approach to assessment of children's mathematics learning.

Tasks

Tasks can be conceptualised in different ways; for instance, MacDonald (2011; 2012) draws attention to the value of mathematical drawing activities and of photographic assignments as tasks for assessing and extending children's understandings at the start of school. In schools, tasks are often initiated by the teacher and this in itself may present a challenge in ensuring that they are meaningful and relevant, and at the very least, motivating and engaging for young children. Educators need to consider the structure and characteristics of tasks and how these relate to the

learning (e.g., Yelland & Kilderry, 2010). Tasks can be teacher-designed or they may be pre-designed ones that accompany curriculum materials. The key issue is that the teacher can identify the possibilities in the child's responses. Guidelines in relation to the use/development of tasks are presented in Report No. 18 (Chapter 2, Section: *Cognitively Challenging Tasks*).

Interviews

Interviews, or focused conversations, are opportunities to explore indepth children's thinking and reasoning through conversation (and observation), generally about tasks that the child undertakes as part of the interview. Observations, tasks and conversations during the course of an interview are methods that complement each other and they are frequently used together (e.g., Ginsburg, 1997b; NRC, 2009). The success of each is contingent on the teacher's knowledge and understanding of early childhood mathematics development (e.g., NRC, 2009). Some curricula in the United States, for example *Big Math for Little Kids* (e.g., Clements & Sarama, 2009b) and *Building Blocks* (e.g., Ginsburg, 2009b), have provided protocols for this work.

Ginsburg advises teacher interviewers to 'adopt, at least provisionally, a theoretical framework with which to interpret your observations' (1997, p. 120). Recently, he discussed how cognitive science can provide that framework in the shape of developmental trajectories or learning paths (Ginsburg, 2009b). He argues that understanding these provides a useful background to understanding individual children. But he also draws attention to the paradox of using developmental trajectories in interviewing:

> *The interviewer's goal is sensitivity to the child. The interviewer wants to have an 'open mind' in order to discover what is in the child's mind. The goal is to learn how the child thinks and how the child constructs a personal world…On the other hand, to discover something about the child's cognitive construction, the interviewer must have some ideas what to look for, some notions about the forms children's thinking may take. Lacking concepts for interpreting the child's behaviour and explanations, the interviewer is likely to overlook what is important and to focus on what is trivial.* (pp. 119–120)

As the educator engages with the child, assessments can be made: of performance, of thinking/knowledge, of learning potential, and of affect/motivation. The information derived can then be used to shape instruction 'in a principled way' (Ginsburg, 2009b, p. 111). The interview, well done, can detect strengths and weaknesses that otherwise may go undetected, but the ability to do the work well is predicated on well-developed mathematical as well as pedagogical subject knowledge. Mathematical knowledge for teaching is discussed in Report No. 18 (Chapter 6, Section: *Mathematical Knowledge for Teaching (MKT)*).

The understanding of the child's perspective, which is elicited in the course of the interview, provides a critical counter-balance to age/stage/level-related presentations of children's

mathematical thinking and acknowledges the child as capable, knowledgeable, logical, sense-making and agentive. It recognises children as competent participants in their education (e.g., Dunphy, 2012). The interview is an opportunity for educator and child to co-construct mathematical understandings. Other significant gains are identified. For example, experience of the interview 'engages the child in talking about one's thinking, justifying one's conclusions, and in general engaging in mathematical communication' (NRC, 2009, p. 264). Ginsburg (1997b) too points to metacognitive and expressive gains: 'the child sharpens, or even acquires the ability to introspect and express thinking' (p. 114). These claims relate to the learning that can happen in the course of an assessment, what Wiliam (2007, p. 1054) refers to as assessment as learning. Because of its sensitivity to the individual, interviewing is particularly useful in seeking to accommodate a diverse range of mathematical abilities.

Conversations

While educators might quickly grasp the benefits of one-to-one interviewing, research has identified a particular need to provide educators with extensive curriculum guidance in interviewing for the purposes of promoting children's mathematics learning (NRC, 2009) often in the context of professional development (Ginsburg, 1997b). In reality, given the busy nature of classroom life, many educators may plan to use an extended interview on only a few occasions in any given year. Focused conversations may be the method of assessment used much more frequently. This method of assessment assumes knowledge of learning paths in different mathematical domains. While educators need to learn to use the interview as a means of making in-depth assessments of a child's understanding of a particular concept or big idea such as counting, more usually teachers also need to have mathematical conversations with children during the course of classroom activities as the opportunity occurs. For example, the child's understanding of shape can be ascertained in the course of activities with blocks or tangrams. Sensitive questioning and the use of a variety of questioning techniques is an area of general pedagogical knowledge that has been highlighted as a key factor in promoting early learning generally (e.g., Siraj-Blatchford et al., 2002). Donaldson's (1984) work illustrated the dramatic effect of the inclusion or omission of a single adjective in questioning children on so-called 'logical' tasks. Furthermore, it is essential that in questioning the youngest children we note her caution that 'the young child…first makes sense of situations (and perhaps especially those involving human intentions) and then uses this kind of understanding to help him make sense of what is said to him' (p. 59). We know that questioning isn't the only way, nor necessarily the best way, of eliciting responses from young children (e.g., Fisher 1990; Norman 1993). The *Aistear* guidelines (NCCA, 2009b) identify a range of methods which the educator can use in interactions with young children. These include naming and affirming children's actions and behaviours; supporting participation and learning, and assisting learning. Interactions such as these present contexts for assessing early mathematics learning.

Pedagogical Documentation

Pedagogical documentation, the documentation of children's learning, is a framework for assessment which originated in the work of the Reggio Emilia preschools in Italy. Learning moments are captured usually through observation, transcription and visual/audio representations such as photos and recordings. This is the content of the pedagogical documentation. What makes pedagogical documentation different to traditional observation is the process that takes place in the collaborative negotiation and revisiting of the learning. Pedagogical documentation may be defined as:

> *both content and process involving the use of concrete artefacts in the form of audio recordings, photographs, examples of the children's work, and collaborative revisitation, interpretation, and negotiation by the protagonists (children, teachers and parents) to promote dialogue and reflection.* (MacDonald 2007, p. 233)

While the approach seems theoretically to have great potential, there are few if any published examples of its use in the area of mathematics learning and teaching. In the study reported here, it proved challenging for teachers working with early literacy in Canadian schools due to the need for high levels of teacher support. However, the examples of the documentation process offered from the Reggio perspective do include some mathematically focused work, for example Shoe and Meter. On the Reggio Children website the project is described thus:

> *The starting point is a concrete request: the school needs a new table. Teachers propose to children to take care of it: what to do? The first approaches to the discovery, to the function and the use of measures. Children have access to the mathematical thinking through the operations of orientation, play, choice of relational and descriptive languages. (http://www.reggiochildren.it/?libro=scarpa-e-metro&lang=en)*[9]

Supporting Children's Progression with Formative Assessment

In rural and regional Australia, research aimed at investigating early childhood educators' thoughts on young children's mathematical thinking and development found that, while preschool teachers were learning and keeping records in relation to mathematics, it didn't extend beyond observation. Participants in that study also reported reluctance to introduce technology into the settings and this was due to their lack of confidence and competence (Hunting et al., 2013). This is significant given the importance that Carr and Lee (2012) accord to technology in identifying and documenting early learning (see discussion of methods above). It is quite likely that similar attitudes are to be found amongst the educator population here. In the United States, the NRC report (2009) notes that while

9 Reflecting on the work of Castagnetti, M. & Vecchi, V. (1997) *Shoe and meter.* Reggio Emilia, Italy: Reggio children.

formative assessment shows great promise, the methods of assessment have not been clearly linked to the teaching that takes place subsequently. A number of mathematics educators suggest that some of the challenges of integrating learning, teaching and assessment can be met by reference to learning and teaching paths. Ascertaining children's learning and their multi-path learning trajectories enables educators to make judgements regarding how best to support future learning. For instance, Ginsburg (2009b) argues that the rich information gained from one-to-one interviews, which may include insights into children's experiences with aspects of mathematics in everyday situations, actually reveals a great deal about children's understanding of mathematics and this information can be used to compile a profile of the child as a mathematics learner. The teacher can then design appropriate learning experiences for the child. As Ginsburg describes it, the teacher can do so since he/she is now in a position to decide 'on a specific course of action with a specific child' (p. 125). In other words, the teacher is now in a position to decide on a teaching path to help the child. Ginsburg sees the teacher's judgement at this point as critical and one that cannot be replaced by a pre-designed script. As he sees it '…the task of teaching mathematics is so complex that a detailed script is likely to do more harm than good' (p. 126).

From a RME perspective, mathematics educators have identified intermediate steps for trajectories in the areas of number, measure and geometry as guidance for assessment. They argue that these ensure that teachers know what to look for (van den Heuvel-Panhuizen, 2008; van den Heuvel-Panhuizen & Buys, 2008). Learning-teaching trajectories as a basis for assessment are discussed further in Report No. 18 (Chapter 3, Section: *Content Areas*). Earlier, we discussed the role of developmental progressions as a support for teachers in assessing learning (see *Chapter 5*). Young-Loveridge (2011) describes how, in New Zealand, individual diagnostic assessments (based on interviews), in conjunction with a research-based framework outlining the learning progression in number, have provided a powerful means for teachers to determine children's starting points and make decisions about ways to enhance learning.

A project which sought to improve mathematics and numeracy outcomes through a sustained, collaborative programme of professional development and action research was carried out in 2004 in South Australia. As part of that project, Perry, Dockett and Harley (2007) worked with preschool educators who engaged in writing learning stories which focused on children's 'powerful mathematical ideas' (see *Chapter 4*). They did so in the context of eight developmental learning outcomes for children's learning in the preschool year as presented in *The South Australian Curriculum, Standards and Accountability Framework* (Government of South Australia, 2001). The findings established the technique of learning stories as a valid assessment method compatible with the holistic approach inherent at the preschool level. The researchers describe how this was achieved through the educators' use of a numeracy matrix. The matrix constructed by the researchers and the educators consisted of 56 cells (8 developmental learning outcomes x 7 powerful mathematical ideas), with each cell of the matrix providing examples of pedagogical questions for the educators as they were teaching towards, assessing or reporting on the developmental learning outcomes. For instance, in relation to the powerful mathematical idea of *Algebraic Reasoning* and the developmental learning outcome that

Children develop a range of thinking skills, the questions generated were 'How do we encourage children to use patterns to generate mathematical ideas?' and 'In what way do we provide opportunities for children to reflect upon their mathematical pattern making?'.

The matrix proved to be a powerful tool for enabling mathematically-focused assessment practices. It appeared that the educators used the matrix as a framework for reflecting on and identifying children's mathematical learning. They used it as a scaffold with which to build the analyses of children's activities, and subsequently to write the resulting learning stories. Significantly, the stories captured both dispositional and content-related learning, and documented learning in relation to both. As observed by the researchers:

> *the matrix is a dynamic reflection of the knowledge of the educators using it, and, as such, should be expected not only to be grounded in the contexts in which these educators work but to change as their knowledge grows.* (Perry, Dockett & Harley, p. 5)

The methods reviewed above will undoubtedly prove challenging for teachers. Nevertheless, if there is to be coherence between mathematics curriculum, pedagogy and assessment, it is clear that educators will need to be supported and encouraged to move towards implementing such approaches in assessing early mathematics learning.

Arising from the above discussion, there are three important themes evident in relation to assessment of early mathematics. They are as follows:

- The role of strong conceptual frameworks such as general developmental progressions when assessing. These determine what teachers recognise as significant learning, what they take note of and what aspects of children's activity they give feedback on.
- The possible benefits of co-constructing assessment with children.
- The potential of digital technologies for documenting learning and for shaping learner identities.

Diagnostic and Summative Assessment

There is considerable debate in the literature on the value of administering more formal measures of early mathematical knowledge, whether those measures comprise screening/diagnostic tests designed for small groups or individuals that are administered using standardised procedures, but mainly produce qualitative information, or more formal standardised tests which are administered to larger groups and almost always lead to norm-referenced interpretations. Indeed, standardised testing in particular has generally been rejected by early childhood educators as a valid assessment approach for use with young children. This position is encapsulated in the following statement by Fiore (2012):

> *In current early childhood classrooms, most assessment is designed to acquire information that will help responsible individuals make decisions in the interest of the child's growth and development. Testing as part of such assessment takes time and resources…This mandatory time either reduces classroom time for free play and exploration or must be carved out of other organized periods of the day…The assessment process is further challenging because teachers recognize that one particular test or score does not paint a full, clear picture of a complex, developing child. This is supported by research that states that standardized testing of children under the age of 8 is scientifically invalid and contributes to detrimental labelling and can permanently damage a child's educational future…* (p. 5)

A key consideration in relation to such tests concerns the aspects of early mathematics measured (that is, what, according to the test, constitutes mathematical knowledge). According to Smith-Chant (2010b), early numeracy tests often measure skills found on the mathematics curricula taught in the early primary years, and may afford limited attention to important preschool numeracy skills that may be foundational for later mathematical development. Such tests may overestimate the formal aspects of numeracy knowledge, particularly in the areas of number-language and arithmetic, and under-estimate the non-language-based aspects of numeracy understanding (e.g., the concept of non-verbal counting, more, less, time and patterning). Moreover, they may have a heavy language component, presupposing that a child's understanding of early numeracy is language-based.

Snow and Van Hemel (2008) outline some key issues than can arise in administering direct assessments such as diagnostic tests and more formal standardised tests. These include the following:

- The child may not be familiar with this type of task or be able to stay focused.
- Young children have a limited response repertoire, being more likely to show rather than tell what they know.
- Young children may have difficulty responding to situation cues and verbal directions.
- Young children may not understand how to weigh alternative choices, for example, what it means for one answer to be the 'best' answer.
- Young children may be confused by the language demands, such as negatives and subordinate clauses.
- Young children do not respond consistently when asked to do something for an adult.
- In some cultures, direct questioning is considered rude.
- The direct, decontextualised questioning about disconnected events may be inconsistent with the types of questions children encounter in the classroom.
- Measurement error may not be randomly distributed across programmes if some classrooms typically use more direct questioning, like that found in a standardised testing situation.

Berliner (2011) argued that many young children may have a restricted ability to comprehend the formal, spoken instructions required for many standardised tests, that they lack the sophistication to interpret situational cues or written instructions, and that a test administered at one point in time may not capture important shifts in changes in a child's development.

While it is accepted that diagnostic and summative assessments may not be appropriate or desirable for use with young children, we recognise that there are contexts in which their use may be seen as helpful (for example, to identify children who may be at risk of learning difficulties). The key issue here is that, if used at all, they should be used as only one measure of children's mathematics learning and development. Next, we consider the types of information that screening/diagnostic tests and standardised tests can provide.

Screening/Diagnostic Tools

The primary purpose of screening/diagnostic tests is to identify children's learning difficulties in mathematics at an early stage, with a view to providing early intervention. Such tests are often administered on a one-to-one basis, allowing test administrators (usually teachers) to evaluate children's responses to set tasks, including the reasoning behind those responses. In relation to the lowest-achieving children, four components of number competence have been highlighted as important to include in screening/diagnostic measures. These are (i) magnitude comparison, or the ability to discern which number in a set is greatest, and relative differences in magnitude; (ii) strategic counting, defined as the ability to understand how to count efficiently and use counting strategies; (iii) ability to solve simple word problems; and (iv) retrieval of basic arithmetical facts (Gersten et al., 2012).

The following are issues that may arise in the administration and interpretation of screening/diagnostics tests, such as *the Drumcondra Tests of Early Numeracy* (ERC, 2011) and the *Learning Framework in Number* (LFIN) (Wright, Martland, & Stafford, 2006):

- Such tests are generally administered to children deemed to be at-risk of learning difficulties in mathematics; hence, not all children in a group will need to be assessed using these methods; indeed many screening/diagnostic tests are not designed to provide detailed information on the abilities of average or higher-achieving learners.
- Screening/diagnostic tests can provide valuable qualitative (formative) information on the reasons underlying children's responses, if test administration allows users to gather and record such information.
- Such tests are often linked to instructional programmes or interventions. In some cases, the interventions have been demonstrated to be effective in a range of contexts; in other cases, there may be limited evidence to support instructional recommendations, and hence care will need to be exercised in deciding what support to provide.

- Performance on screening/diagnostic tests (and on other types of tests) may be associated with factors such as educational disadvantage or the children's linguistic skills, and these factors need to be taken into account in interpreting outcomes.
- Performance on screening/diagnostic tests can be predictive of later performance on more formal standardised measure of mathematics (e.g., ERC, 2011). However, such tests may not be predictive at the individual child level, and other evidence, in addition to the outcomes on a screening/diagnostic test, may need be taken into account in making inferences about a child's risk status.
- Screening/diagnostic tests for young children often focus on number, and other important aspects of numeracy or mathematics, such as shape and space, may be overlooked.

Standardised Norm-Referenced Tests

In general, group-administered, standardised tests of numeracy or mathematics are deemed inappropriate for use with young children. Indeed, in the US, states are not required to administer standardised tests for accountability purposes until children are in the latter part of third grade (8–9 years of age). Similarly, while assessment at Key Stage 1 in England originally comprised formal paper-and-pencil tests in mathematics, this is no longer the case, and teachers now submit results based on their own professional judgements, though supports are available to help teachers make judgements, including optional tests.

In Ireland, standardised tests are administered to children in second class, as part of the National Assessments of Mathematics Achievement (see Eivers et al., 2010), which is conducted every five years. In addition, since 2012, schools are expected to administer standardised tests to children in second, fourth and sixth classes, and to report the outcomes to parents and to the school's Board of Management. Schools may exempt certain children from testing and/or reporting, though criteria for this are not well defined. Drummond's (2012) analysis of the test performance of a young boy named Jason (aged 7 years 6 months), provides a graphic account of the inadequacies in using such tests with children of this age as a way of assessing individual learning.

While standardised tests can provide an overall indication of a child's performance (for example, a standard score, percentile rank or sten score), and some of these scores can be aggregated across children at the same class level (e.g., the proportions of children in a class scoring at each sten score), they provide limited diagnostic information, and, where such information can be generated, it may be distorted because the tests have to serve multiple purposes.

Although standardised tests are typically based on a framework that broadly mirrors the underlying curriculum, there may be limited value in relying on content-area or process subtest scores. This is because, in general, there may be too few items on a subtest to allow for reliable information to be generated. This often tends to be the case with the Data strand, which may be represented by just a few items on a test for young children.

The recent increased emphasis on standardised testing for accountability purposes (e.g., DES, 2011) may also lead to an increased emphasis on preparing children to take standardised tests (meaning that test content becomes very familiar to children over time). A consequence of this is that test performance may improve, but children's proficiency in mathematics may not change.

Finally, standardised tests do not provide information on such factors as procedural fluency (accuracy, efficiency and flexibility), strategic competence, adaptive reasoning (logical thinking and justification) or productive disposition (behavioural-emotional components) (Mueller, 2011). In other words, standardised tests tell us nothing about these key strands of mathematical proficiency. Clearly, where used, standardised tests can only be considered to comprise one element of a more comprehensive assessment framework for planning, teaching, and learning of mathematics and which has at its centre a strong practice of formative assessment.

Many of the issues that arise in administering and interpreting the outcomes of group-administered standardised tests also apply to individually-administered standardised tests. These include the range of mathematical knowledge assessed and the lack of information on children's thinking processes. However, an individually-administered test does allow for the creation of an easier rapport between test administrators and child, than is possible with a group-administered test.

Planning for Progression Using Assessment Outcomes

A primary purpose of gathering assessment information is to use it as a basis for planning instruction. Where the mathematical development of young children including preschoolers is concerned, adults will need to draw on the outcomes of appropriate forms of formative assessments – observations, tasks, interviews and conversation. The interpretation of outcomes is guided by the adult's understanding of children's general cognitive development (what should be expected at different developmental points in terms of language and understanding), as well as mathematical development (e.g., through familiarity with learning paths – see above). As outlined in *Aistear* (NCCA, 2009b), a key aspect of assessment is the recording of assessment data so that adults have a basis on which to plan future learning activities, taking into account children's current knowledge and their needs.

Planning for progression will occur at the level of the individual teacher/carer, and among groups of adults working with or at least familiar with the same children. The literature (e.g., Ginsburg, 2009a) suggests that, for younger children, the focus is on

- the mathematical knowledge that children bring from home (including invented strategies) and how this relates to opportunities for mathematical development presented in preschool/early primary school
- the quality of their everyday language and their mathematical language, including their knowledge and use of key terms in areas such as number and shape and space

- their ability to talk about problem-solving in formal and informal mathematical activities
- their understandings and metacognition with respect to mathematics – their sense of their own ability to solve a mathematical problem
- their abilities to make connections across aspects of mathematics, and between mathematics and everyday life
- their mathematical dispositions.

As children progress through the primary school classes, teachers may extend the range of assessment outcomes that they use in their planning to include those arising from screening/diagnostic tests, and, perhaps towards the end of the 3–8 years range, from standardised tests. At this stage, it is important to integrate the outcomes of formative and diagnostic/summative assessments since, as Ginsburg (2009b) points out, standardised tests, in and of themselves, do not provide information about children's underlying thinking processes. In this view, children might do quite well on a standardised measure, yet may lack the sense-making and critical thinking that are the hallmarks of mathematical proficiency.

Immersion Settings

One immersion setting in the Irish context is the Irish-medium setting – whether naíonraí[10] or primary schools – where children may learn mathematics in a first, second, or third language – Irish. Here, teachers will have to take children's proficiency in Irish into account in interpreting assessment outcomes – does the child have sufficient language proficiency to understand the task being assigned, and to express his/her mathematical thinking (see also Chapter 3, Section: *Variation in Language Skills and Impact on Mathematics*). There is, for example, evidence from the 2010 national assessments of English and mathematics in Irish-medium schools (Gilleece et al., 2012) that children may have struggled with the language on a standardised test of mathematics administered in Irish, and hence may have performed less well than they were capable of. This, perhaps, underlines the importance of combining information from multiple sources in arriving at inferences about the mathematical performance of children in such settings.

Wood and Coltman (1998) argue that 'it is difficult to over-emphasise the importance of verbal communication in the development of children's mathematical understanding' (p. 114). The implication of this is that empowering children to develop their language skills in the language of instruction of the school is of vital importance for supporting the development and expression of mathematical understanding. Many children in Irish-medium settings have a common language, English, shared among themselves and the teacher. This facilitates communication, even though the

10 Naíonra is a playgroup run through Irish for children aged 3–5 years, who attend daily for 2–3 hours.

language policy of the settings may be discouraging of this. Code-mixing, where utterances involving vocabulary or structures from two or more languages are combined, is often used as a strategy to allow communication and understanding (Mhic Mhathúna, 1999).

Similar issues arise in addressing the assessment needs of children whose first language is not English or Irish. Commentators differ on the need to provide assessments for the child in their stronger language (Baker, 2001), or in the language of instruction (Sierra, 2008). Peal and Lambert (1962) established that proficiency in both languages resulted in higher scores in verbal and non-verbal testing of intelligence, an early forerunner to Cummins' threshold theory (Cummins, 1976; Cummins 2000). The threshold theory suggests that bilingual children who have achieved a level of competence in both languages are afforded a cognitive advantage in all other areas of the curriculum. Conversely, children who have not reached a minimum standard of competence in both languages may experience negative cognitive and academic outcomes, with obvious implications for mathematics learning. While the threshold theory has been criticised for failing to clarify in concrete terms what these thresholds are (Chin & Wigglesworth, 2007), or for equating academic success with cognitive ability, without allowing for factors such as socio-economic status (MacSwan, 2000), evidence to support it has emerged from the US (Kessler & Quinn, 1982), Ireland (Ní Ríordáin & O'Donoghue, 2007), Malta (Farrell, 2011), and Papua New Guinea (Clarkson, 1992).

This suggests that age-appropriate levels of language competence in both languages should be considered when forming assessment opinions of children's achievements in mathematics. Educators carrying out assessment procedures such as interviews, observations or tasks in an immersion context have the dual purpose of assessing and evaluating both the mathematical competences and language competences of the child to gain a full picture. Dual language assessment (Murphy & Travers, 2012) is particularly important in this context, though it should be recognised that this adds to the complexities of the process, and to the demands on the child. When developing assessment materials or guidelines in a dual language context, care needs to be taken to ensure that tasks or questions on both language forms are developed collaboratively by translation and education experts to ensure their validity in both languages and minimise the danger of dealing with unfamiliar vocabulary or language constructions in either language, which would hinder the expression of mathematical knowledge or thinking on the part of the child (Rogers, Lin & Rinaldi, 2011). It might also be considered that difficulties that immersion children experience in mathematics may best be addressed not only by interventions aimed at supporting mathematical concept and skill acquisition, but also by interventions aimed at raising general language competence in both languages.

Children with Special Needs

The assessment of the mathematical and other abilities of children with special educational needs is complex. According to Snow and Van Hemel (2008):

- It is important to use multiple sources of information in arriving at decisions about the needs of children with special education needs, as the performance and behaviour of children with special needs across settings and situations can be even more variable than those of typically developing children.
- The variability in the performance of children with special needs across situations requires incorporating information from family members to obtain an accurate picture of the child's capabilities.
- A key principle applicable to all children but of special relevance to children with special needs is the importance of providing them with multiple opportunities to demonstrate their competencies.
- The setting for the assessment, the child's relationship with the person conducting the assessment, the ability of the assessor to establish rapport, fatigue, hunger, interest level in the materials and numerous other factors could result in a significant underestimation of the child's capabilities.
- Many young children with special needs are not capable of complying with all of the demands of testing situations, arising from lack of language, poor motor skills, poor social skills, and lack of attention and other self-control behaviours.
- Assessment tools should have a low enough floor to capture the functioning of children who are at a level far below their age peers.
- In assessing young children with special needs, it is important to consider the test's assumptions about how learning and development occur in young children and whether these are congruent with how development occurs in the child being assessed.

Therefore, considerable care needs to be exercised in the use of formal approaches to assessment with young children with special needs. In such circumstances, appropriate formative assessment methods may present the best solution. In relation to this, Douglas et al. (2012) note that, on occasion, child-led assessment through conversation methods may be problematic. It has also been suggested that, for some children with special needs, attentiveness should also be a focus of assessment (e.g., Gersten et al., 2012), as these children may not have the attention required to concentrate on the task in hand.

Conclusion

In relation to assessing and planning for progression in children's mathematical development, a number of approaches were reviewed, including the use of formative, diagnostic and summative assessments. Formative assessment methods are seen as coherent with the image of children as active powerful learners who learn mathematics as they engage in everyday activity with parents/caregivers, peers and teachers. Screening and diagnostic approaches are seen as useful for recognising and supporting

children who are having difficulties with mathematics. Attention is drawn to the inappropriateness of standardised tests and their inability to adequately portray the mathematical learning and development of young children.

The key messages in this chapter are as follows:

- Of the assessment approaches available, formative assessments offer most promise for generating a rich picture of young children's mathematical learning.
- Strong conceptual frameworks, including a sound understanding of general developmental progressions (learning paths), are important for supporting teachers' formative assessments. These determine what teachers recognise as significant learning, what they take note of and what aspects of children's activity they give feedback on.
- There is a range of methods (observation, tasks, interviews, conversations, pedagogical documentation) that can be used by educators to assess and document children's mathematics learning and their growing identities as mathematicians. These methods are challenging to implement and require teachers to adopt particular, and for some, new, perspectives on mathematics, on mathematics learning and on assessment. Digital technologies offer particular potential in this regard.
- Constructing assessments which enlist children's agency (for example, selecting pieces for inclusion in a portfolio or choosing particular digital images to tell a learning story) has many benefits, not least of which are the inclusion of children's perspectives on their learning and their assessments of their own learning.
- More structured teacher-initiated approaches and the use of assessment within a diagnostic framework may be required on some occasions, for example, when children are at risk of mathematical difficulties.
- The complex variety of language backgrounds of a significant minority of young children presents a challenge in the learning, teaching and assessment of mathematics. Children for whom the language of the home is different to that of the school need particular support in developing language in order to maximise their opportunities for mathematical development and their participation in assessment.
- Educators carrying out assessment procedures such as interviews, observations or tasks in an immersion context have the task of assessing both the mathematical and language competencies of the child to gain a full picture of their development. Dual language assessment is particularly desirable in this context. This applies to both EAL and to Irish-medium settings.

CHAPTER 7

Addressing Diversity

In line with the inclusive nature of the perspective adopted in this report, it is important to reiterate the assumption that mathematics is relevant to all children and that each child has the right to access, participate in and benefit from enriching mathematical experiences. In discussing the literature and perspectives on children who experience difficulties learning mathematics, there can be a disproportionate emphasis on gaps and needs. However, we would preface this chapter by stressing that all children have strengths and preferences in relation to mathematics and that the goal is always to support the child through using these strengths and preferences. It is also important to understand what we do not know in relation to mathematical development and take the perspective that any perceived difficulties and delays are the responsibility of the teacher and school to address. The groups of individuals who often require particular attention in the teaching and learning of mathematics are 'exceptional' children (those with developmental disabilities or who are talented mathematically), children for whom English is not a native language or those living with disadvantage. In this chapter, an overview is given of the different ways exceptional children are grouped and how attention might be given to their particular needs. Some consideration is also given to addressing cultural diversity in mathematics learning. In essence, it is to be argued that mathematics teaching that is sensitive to and appreciative of individual and/or group variation is effective 'mathematics for all'.

Identification of Learning Difficulties in Mathematics

Butterworth (2005) claims that 'specific disorders of numeracy are neither widely recognised nor well understood' (p. 12). Attempts to categorise and label children experiencing low achievement/ learning difficulties in mathematics have been problematic. Such approaches underestimate the role of instruction and experience in the development of critical knowledge and skills. It often assumes that, because children are in the same class with the same teacher, this can be controlled for. However, a myriad of influences affects how children construct knowledge and interact and engage

with a teacher and their environment, or not. Also, it is very difficult to isolate the influence of inappropriate teaching or home and preschool experiences on low achievement in mathematics. Data on low achievement often do not distinguish between a delay, a temporary difficulty and more persistent long-term difficulties in the subject. Given this uncertainty, Dowker (2004) recommends that 'ultimately, the criteria for describing children as having 'mathematical difficulties' must involve not only test scores, but the children's educational and practical functioning in mathematics' (p. i).

A response to the above uncertainty with definitions and criteria in the US has been the development of the Response to Intervention (RTI) initiative. This arises out of the importance of monitoring the effectiveness of mathematical teaching and learning prior to classification of a learning disability. The primary goal of RTI is the prevention of difficulties through tailored evidence-based interventions. A secondary goal is the use of the data on progress with the intervention for the referral and identification of students with specific learning disabilities. It is now part of the US federal law in this area.

Exceptional Children

Kirk et al. (2012) define as 'exceptional' a child who differs from the 'typical' child in (i) mental characteristics, (ii) sensory abilities, (iii) communication abilities, (iv) behaviour and emotional development and/or (v) physical characteristics. The term includes both the child with developmental delays and the child with gifts and talents. In their view:

> *Individuals with exceptionalities help us better understand human development. Variation is a natural part of human development; by studying and teaching children who are remarkably different from the norm, we learn about the many ways in which children develop and learn. Through this knowledge, we inform ourselves more thoroughly about the developmental processes of all children.* (p. 3)

They remind us that the term 'typical' is problematic (that is, each of us differs from others in some regard) but, from an educational perspective, 'exceptional' usually suggests a learner for whom some modification has to be made to accommodate his or her individual needs. Notwithstanding this, there is broad consensus that 'distinctive teaching approaches' are not required for exceptional learners, although there is a need to address individual needs (e.g., Davis & Florian, 2004).

Intellectual and Developmental Difficulties

In reviewing the literature, distinctions are made between children with specific difficulties in mathematics, and children with difficulties with components or sub-components of mathematics. In addition, children can have difficulties arising from or as a risk factor because of a disability, specific or general.

Specific Difficulties in Mathematics

Difficulties in learning mathematics have been recognised for at least a century (Siegler, 2007). Multidisciplinary research of the issues has increased in recent decades but it lags substantially behind the equivalent level of attention afforded to literacy difficulties. Likewise, the evidence base is not as strong as for reading. However, there is consensus that a significant number of children exhibit poor achievement in mathematics (Swanson, 2007).

In studying the nature of difficulties in mathematics, Dowker (2004) emphasises the crucial understanding that arithmetic is not a single entity but is made up of many components. By arithmetic ability is meant

> *knowledge of arithmetical facts; ability to carry out arithmetical procedures; understanding and using arithmetical principles such as commutativity and associativity; estimation; knowledge of mathematical knowledge; applying arithmetic to the solution of word problems and practical problems; etc.* (p. ii)

Studies highlight that it is possible for children to show marked discrepancies between components of arithmetic. Dowker (2004) concludes that 'children, with and without mathematical difficulties can indeed have strengths and weaknesses in almost any area of arithmetic' (p. 5). Despite this variability, research has pinpointed some areas of mathematics that create more problems for children than others, though there is less agreement on the underlying mechanisms underpinning these patterns.

There is much written on the nature of the difficulties that children can have and comparisons with children without such difficulties. Siegler (2007) highlights a number of promising developments in the field regarding the structure of mathematical disabilities: Geary et al. (2007) highlight the role of three processes: working memory functioning, phonological processing and visuo-spatial thinking; Butterworth and Reigosa (2007), working from the perspective of neuro-imaging, which often shows different results to other approaches such as interviewing, suggest that domain-specific modular representations of number play a role, and that there is little evidence supporting the role of working memory. They suggest that children at risk of mathematical difficulties are slower at subitising (saying how many are in a small set without counting). An increasing number of researchers such as Jordan (2007), Barnes et al. (2007) and Bull (2007) emphasise the role of poor mastery of number facts and fact retrieval, poor number sense and weaknesses in conceptual understanding as underlying problems. In addition, researchers highlight social and emotional influences such as motivation and maths anxiety.

Developmental Delay

Berch and Mazzocco (2007) make a distinction between children with developmental delay and those with a mathematical learning disability. Dowker (2004), in a review of the area, highlights that

> *a significant number have relatively specific difficulties with mathematics. Such difficulties appear to be equally common in boys and girls, in contrast to language and literacy difficulties which are more common in boys.* (p. i)

This raises the question as to the differences between children with specific difficulties with mathematics and those with non-specific difficulties associated with low achievement in general. There has been inconsistency in the findings related to this question but more evidence is pointing to specific difficulties being milder than difficulties associated with low achievement in general.

Children with certain disabilities can experience difficulties in mathematics. Research has highlighted, for example, difficulties for children with specific language impairment (Donlan, 2007); Turner and fragile X syndromes (Mazzocco et al., 2007); spina bifida (Barnes et al., 2007); attention deficit hyperactivity disorder (Zentall, 2007) and with brain injuries (Zamarian et al., 2007). Mathematical difficulties often co-occur with dyslexia and language difficulties (Dowker, 2004). While children with some forms of brain damage or genetic disorder can have disproportionate difficulties with number, on the whole, children with general learning disabilities display similar developmental profiles as peers of the same mental age (Dowker, 2004).

Porter (1999) makes the distinction between what children can do and what they understand. In a study comparing the performance of children with severe learning disabilities and nursery-school (i.e., preschool) children, Porter (1998) found no difference in performance on simple counting and error-detection tasks. However, there was a difference in the acquisition of counting skills. Porter outlines four profiles of performance: non-counters, acquirers, transitional, and error detectors. Mental age proved to be the best predictor of performance tested. The pattern of attainments of the children described as acquirers differed from that of preschoolers in that adherence to the one-to-one principle was easier than adherence to the stable-order principle for both small and large sets. The performance of the children suggested that it was necessary to learn the skills of counting prior to understanding what it means to count.

In a review of the literature on deaf children and mathematics learning, Nunes (2004) concludes with a hypothesis that 'deafness is a risk factor for difficulties in learning mathematics rather than a cause' (p. 151). In addition, findings suggesting that lack of informal mathematical experience may have a 'wide-ranging effect on deaf children's logical and mathematical development' (p. 155) are presented. In experiments in problem-solving, Nunes found 'that there is a gap between hearing and deaf children's use of actions to solve problems and that this gap is often more severe when the actions have to be coordinated with counting' (p. 154). Marschark and Spencer (2009) conclude that:

> *Delays in language development, a relative lack of exposure (incidentally and in classrooms) to life-based problem-solving activities, and frequently inadequate pre-service teacher preparation in mathematics are believed to lead to the overall delay in development of maths concepts and skills by students with hearing loss. Below-age language skills limit access to teacher-provided as well as text-based explanations and most deaf and hard-of-hearing students lack age-appropriate command of technical vocabulary in mathematics.* (pp. 139–140)

Mathematically Talented Children

In TIMSS 2011 mathematics, in which fourth class children in over 50 countries participated, the percentage in Ireland reaching the Advanced International Benchmark, while twice the international median, was well below the percentages in the top three performing countries (Singapore, the Republic of Korea, and Hong Kong), and also well below the percentages for Northern Ireland and England (Eivers & Clerkin, 2012). This suggests that many children in Ireland may not reach their potential in mathematics, compared with their counterparts in high-scoring countries. There is broad consensus that, internationally, the needs of children who are advanced (talented) in mathematics are not met (Diezmann et al., 2004). For this reason, these children 'underachieve' and are at risk of becoming quietly disaffected from mathematics in future years (Nardi & Steward, 2003). Such children may, however, be quite advanced in different mathematical domains, e.g., in a capacity to reason analytically or spatially (or perhaps both), and teachers need to be sensitive to the varying needs of these children (Diezmann & Watters, 2002). In particular, the needs of these children can be met by the provision of challenging tasks that have scope for learning and the use of metacognitive skills (ibid). However, this is not a case for 'streaming' or for a differentiated curriculum. Rather tasks can be created that allow all children some form of success. In this regard, Sohmer et al. (2009) speak of tasks with 'high-level cognitive demands'. Such tasks are characterised by 'multiple entry points, solution strategies and interpretive claims' (p. 112) and allow different students to access them in a variety of ways. Fiore (2012) refers to these tasks as 'tiered assignments':

> *The idea behind tiered assignments is to provide students with parallel tasks that have different levels of depth, complexity, and abstractness, as well as different support elements or explicit guidance. All students work toward the same goal or outcome, and the differentiated tasks allow students to build on their prior knowledge and strengths while their work on the tasks provides them with appropriate challenges.* (p. 143)

In redeveloping the mathematics curriculum for 3- to 8-year-olds, consideration needs to be given to the design and development of such tasks.

Cultural Variation

Wright (1994) describes a 3-year difference in the numerical knowledge of children as they begin primary school. Some 4-year-olds have attained a knowledge of number that some of their peers will not attain until they are 7 years old. Griffin et al. (1994), using a standardised test of children's conceptual knowledge, also found a 3-year gap in performance among 5- to 6-year-olds, with children from low-income communities performing like middle-income 3- to 4-year-olds. Without any intervention, such gaps widen throughout primary school. The Cockcroft report (1982) found that in a class of 11-year-olds, there is generally likely to be a 7-year range in arithmetical ability.

The general view espoused in this report that a teaching approach that is linked to meaningful cultural referents and that assumes that all children have the capacity to engage successfully in mathematics is an effective approach for all children regardless of their gender, ethnicity or social class (e.g., Ladson-Billings, 1995). It is also assumed that individual variation is the norm and not the exception (Fiore, 2012).

Ethnicity

In the 2009 *National Assessments of Mathematics and English* (Eivers et al., 2010), one of the factors associated with lower child achievement (in both English and mathematics) was that of speaking a first language other than English or Irish. Indeed, children are often perceived to be experiencing difficulty in mathematics on the basis of their relatively poor performance in achievement tests. However, such comparisons – while perhaps useful in terms of highlighting disparities – focus on access and achievement from a dominant perspective (generally white, middle-class students) and thereby preserve the status quo (Gutiérrez & Dixon-Román, 2011). A particular problem is that equity issues tend to be discussed from the perspective of group differences. Secada (1995) puts it like this:

> *[T]he search for group differences grants legitimacy to the view that diverse student populations are somehow deficient, exotic, or primitive when measured against the dominant norm. However, if all one can write or speak about is how a specific group is different from the norm, then the results are an impoverished view of that group and the validation of the belief that equity groups are somewhat inferior.* (p. 153)

Fiore (2012) exhorts the need for teachers to be 'culturally responsive' and decries the effect of test scores on such practices:

> *In an ideal situation, culturally responsive teachers, curricula, and assessments would support children's diversity, but the reality of pressures to produce evidence of annual yearly progress means that children's learning styles, language, temperaments, and identities are viewed as potential obstacles to successful assessment scores and ratings.* (p. 128)

Dooley and Corcoran (2007) argue that, while distinctive teaching approaches for different groups can lead to a deficit view of mathematics learning, the notion of 'one curriculum for all' might also perpetuate inequalities. Malloy (1999) proposes that pedagogy, not content, must become multicultural. This means valuing the many ways that children make sense of mathematics. Some means by which this might be achieved by teachers include ensuring that connections between new and old ideas are evident to the learner, using problem contexts that are meaningful to the child, and focusing on the child's intuitive representations and informal procedures (Carey et al., 1995). This idea of a multicultural pedagogy receives further attention in Report No. 18 (Chapter 4, Section: *Children in Culturally Diverse Contexts*).

Socioeconomic Disadvantage

As noted in the introduction to this report, children in Ireland with low socioeconomic status perform less well, on average, than their more advantaged counterparts. Group differences are most notable among children attending schools in the urban dimension of the School Support Programme under DEIS, with average scores among children attending the most disadvantaged schools (those in urban DEIS Band 1) between three-fifths and four-fifths of a standard deviation below those of children in non-DEIS urban schools in the most recent national assessment (Eivers et al., 2010).

While the national assessments focus on the mathematical performance of children in second and sixth classes, international research indicates that the relationship between socioeconomic status (SES; usually defined in terms of parents' income level and/or education) and mathematics performance manifests itself considerably earlier (NRC, 2009). Pre-verbal number sense, which involves the ability to discriminate between large arrays of various sizes, begins in early infancy and appears to be universal (Xu, Spelke & Goddard, 2005). Preschool and early school number sense, which involves an understanding of number words and symbols, is more heavily influenced by experience and instruction, and large differences in performance are evident by the time children enter preschool, on standardised tests and on measures such as determining set size, comparing sets, or carrying out calculations (Klibanoff et al., 2006). Early differences in mathematical performance between children in families described as middle- and low-SES emerge for spatial/geometric understanding and measures as well as for number competencies (Clements, Sarama & Gerber, 2005). The importance of preschool number sense is underlined by strong correlations between measures of number sense at preschool level and success on mathematics later in childhood (e.g., Fuchs et al., 2007). Young children's number skills can be measured using either verbal tasks (i.e., number tasks without objects) or non-verbal tasks (i.e., number tasks with objects). Children in disadvantaged circumstances often perform less well on the former and their growth rates in kindergarten and first grade tend to be lower, compared with number tasks in which objects are present. Differences have also been observed on the same number tasks when presented verbally and non-verbally, with performance among children identified as disadvantaged being lower in verbal contexts (Jordan et al., 2007).

However, there is also evidence that early knowledge of number words can assist young children on tasks that do not require verbal input, such as matching arrays of visual dots. In a study by Ehrlich et al. (2006), children (aged 2–3 years) identified as low-SES tended to do less well on such tasks, compared with children identified as middle-SES, though differences were eliminated if responses that were plus or minus 1 from the correct answer were accepted as correct. This research suggests that preschool children identified as low-SES have approximations of set sizes and number words, at a time when other children have achieved exact representations.

The US NRC Report (NRC, 2009) suggests that early differences on mathematical tasks among children with differing backgrounds can arise for a number of reasons, including the amount of support for mathematics at home and language and contextual factors. Clements and Sarama (2008) also point out that preschool programmes in the US that serve the most disadvantaged children tend to provide fewer opportunities and support for mathematics development than programmes that support more advantaged children. Jordan and Levine (2009) take the view that weaknesses in number competence can be identified in early childhood and that most children (including the most disadvantaged) have the capacity to develop the number competence that lays the foundation for later learning.

There is a broad range of factors that need to be considered in designing programmes to support the early mathematical development of less-advantaged children. These include

- *Parental beliefs and behaviours.* Parents in general prioritise the development of early literacy and language skills and living skills over mathematical skills (Barbarin et al., 2008) and expect preschools to focus on language and literary skills rather than early number skills (Cannon & Ginsburg, 2008).
- *Nature of parent-child interactions.* Even in studies in which parents involve their less- or more-advantaged children in informal mathematical activities such as talking about number, playing with puzzles and shapes, engaging in counting, and using number symbols to represent quantity, differences in mathematical performance between children with varying degrees of disadvantage can arise, suggesting subtle differences in the effectiveness of parents in differing contexts (Saxe et al., 1987).
- *Language.* There is a wide variation in instances of mathematical language both in homes and in preschool settings (see Chapter 3, Section: *Variation in Language Skills and Impact on Mathematics*), which can impact on children's developing mathematical competence in a range of areas, including number and spatial awareness.
- *Parent expectations.* Studies show a tendency among parents to overestimate their children's mathematical competence in aspects such as cardinality (Fluck, Linnell & Holgate, 2005). This may limit the frequency and intensity of mathematical activities involving young children and their parents.

Differences that characterise the home and community backgrounds of children living in disadvantage underlines the importance of: (a) supporting parents of such children to enhance their children's mathematical competence through engagement in and discussion on a range of mathematics-related activities; (b) ensuring that preschool programmes, as well as building on child-led play and naturally occurring opportunities, include a strong numeracy component that includes opportunities for children to engage in planned activities with varying degrees of structure that expose them to mathematical ideas; and (c) providing opportunities for children in home, preschool and primary school settings to engage in language interactions with adults about important mathematical ideas and symbols, whether during structured play or storytime. Although these activities are often recommended for all children, their frequency and intensity may need to be raised in contexts in which large numbers of children from disadvantaged backgrounds meet together.

There is a range of intervention programmes in place in DEIS schools in Ireland for children who may be at risk of mathematical difficulties, including *Mathematics Recovery (Mata), Ready, Set, Go Maths,* and *Maths for Fun*. While these programmes incorporate many of the goals of effective early years interventions, we could not find any published evaluations of the effects of the programmes on young children's mathematical development in the Irish context. It would seem important to monitor the effects of these programmes, and, in particular, the extent to which skills acquired through early intervention are maintained and extended once children exit from the programmes.

Conclusion

Individual and/or group variation should be regarded as a strength of the educational system and the redeveloped mathematics curriculum needs to address learner variability. It is not that distinctive teaching approaches (or indeed distinctive curricula) are required but that mathematics teaching should address specific needs – including the needs of those who are exceptional because of a disability or talent, those who do not have English or Irish as a mother language or those coming from a disadvantaged background. The implications of this position are immense. In particular, there is a need to move away from over-reliance on data from group testing to inform policy and practice in the area and to supplement group data with other relevant assessment information. Furthermore, teachers need to be supported in the design, development and delivery of mathematics lessons that recognise and capitalise on learner variability. Some of the challenges inherent in this task are explored in Report No. 18 in the discussion on an equitable curriculum (see Chapter 4, Section: *Children in Culturally Diverse Contexts*).

The key messages arising from this chapter are as follows:

- The groups of individuals that often require particular attention in the teaching and learning of mathematics are 'exceptional' children (those with developmental disabilities or who are especially talented at mathematics), children who speak a first language other than English/Irish at home, and children living in disadvantage. In addressing their individual needs, the use of multi-tiered tasks, in which different levels of challenge are incorporated, is advocated.
- Mathematics 'for all' implies a pedagogy that is culturally sensitive and takes account of individuals' ways of interpreting and making sense of mathematics. In particular, norms-based testing can disadvantage certain groups. A diverse range of assessment procedures is required to identify those who experience learning difficulties in mathematics.
- Parents and educators need particular supports in constructing mathematically-interactive and rich environments for children aged 3–8 years. The intensity of the support will need to vary according to the needs of particular groups of children.

CHAPTER 8

Key Implications

The purpose of this report is to inform the redevelopment of the mathematics curriculum for children aged 3–8. In addressing this we focused on research related to the mathematical education of children aged 3–8 years. We drew on a broad range of relevant literature and research studies, particularly those published since the introduction of the current Primary School Mathematics Curriculum in 1999. In line with the research request, we focused on definitions of mathematics education, theoretical perspectives, the role of language and communication in learning mathematics, goals, stages of development, diversity and assessing and planning for progression.

The implications for curriculum development presented here are based on a view of mathematics as useful and as a way of thinking, seeing and organising the world, as well as being aesthetic and worthy of pursuit in its own right (Zevenbergen, Dole, & Wright, 2004). They are based on a view of all children as problem-solvers who can make sense of the world using mathematics, who engage in the processes of mathematization, and who develop productive dispositions towards mathematics.

Our implications are presented in a context in which there is a growing awareness of children's early mathematical knowledge and how it can be developed. Other important contextual factors include the multicultural nature of children's learning environments, the ever-growing presence of technology in all aspects of children's lives, concerns about children's mathematical achievements and attitudes, and an economy in which mathematical knowledge is increasingly valued.

The key implications arising from this review of research presented in this report are as follows:

- In the curriculum, a view of all children as having the capacity to engage with deep and challenging mathematical ideas and processes from birth should be presented. From this perspective, and in order to address on-going concerns about mathematics at school level, a curriculum for 3–8 year-old children is critical. This curriculum needs to take account of the different educational settings that children experience during these years.
- The curriculum should be developed on the basis of conversations amongst all educators, including those involved in the NCCA's consultative structures and processes, about the nature of mathematics and what it means for young children to engage in doing mathematics. These conversations should be informed by current research, as synthesised in this report and in Report No. 18, which presents a view of mathematics as a human activity that develops in response to everyday problems.

- The overall aim of the curriculum should be the development of mathematical proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition). As mathematization plays a central role in developing proficiency, the processes of mathematization should permeate all learning and teaching activities. These include connecting, communicating, reasoning, argumentation, justifying, representing, problem-solving and generalising.
- The curriculum should foreground mathematics learning and development as being dependent on children's active participation in social and cultural experiences, while also recognising the role of internal processes. This perspective on learning provides a powerful theoretical framework for mathematics education for young children. Such a framework requires careful explication in the curriculum, and its implications for pedagogy should be clearly communicated.
- In line with the theoretical framework underpinning the curriculum, mathematical discourse (math talk) should be integral to the learning and teaching process. The curriculum should also promote the development of children's mathematical language in learning situations where mathematics development may not be the primary goal. Particular recognition should be given to providing intensive language support, including mathematical language, to children at risk of mathematical difficulties.
- The goal statements of the curriculum should be aligned with its underlying theory. An approach whereby processes are foregrounded but content areas are also specified is consistent with a participatory approach to mathematics learning and development. In the curriculum, general goals need to be broken down for planning, teaching and assessment purposes. Critical ideas indicating the shifts in mathematical reasoning required for the development of key concepts should be identified.
- Based on the research which indicates that teachers' understanding of developmental progressions (learning paths) can help them with planning, educators should have access to information on general learning paths for the different domains. Any specification of learning paths should be consistent with sociocultural perspectives, which recognise the paths as provisional, non-linear, not age-related and strongly connected to children's engagement in mathematically-rich activity. Account needs to be taken of this in curriculum materials. Particular attention should be given to the provision of examples of practice, which can facilitate children's progression in mathematical thinking.
- The curriculum should foreground formative assessment as the main approach for assessing young children's mathematical learning, with particular emphasis on children's exercise of agency and their growing identities as mathematicians. Digital technologies offer particular potential in relation to these aspects of development. The appropriate use of screening/diagnostic tests should be emphasised as should the limitations of the use of standardised tests with children in the age range 3–8 years. The curriculum should recognise the complex variety of language

backgrounds of a significant minority of young children and should seek to maximise their meaningful participation in assessment.

- A key tenet of the curriculum should be the principle of 'mathematics for all'. Central to this is the vision of a multicultural curriculum which values the many ways in which children make sense of mathematics. While there are some groups or individuals who need particular supports in order to enhance their engagement with mathematics, in general distinct curricula should not be advocated.
- Curriculum developments of the nature described above are strongly contingent on concomitant developments in pre-service and in-service education for educators at preschool and primary levels.

GLOSSARY

Glossary

Abstraction

an idea based on experiences but independent of any one experience (NRC, 2001, p. 110); mathematics is about increasingly being able to deal with ideas rather than events.

Adaptive reasoning/expertise

the capacity for logical thought, reflection, explanation, and justification.

Big ideas

the overarching concepts that are mathematically central and coherent, consistent with children's thinking and generative of future learning (Clements and Sarama, 2007, p. 463);

overarching concepts that connect numerous topics and applications (Baroody et al., 2006, p. 205).

Conceptual understanding

understanding of mathematical concepts, operations and relations.

Context

an event, issue or situation derived from reality, which is meaningful to the children or which they can imagine and which leads to using mathematical methods from their own experience. It provides concrete meaning and support for the relevant mathematical relations or operations. Situations might be drawn from everyday experiences such as bus rides, or shopping, and handling money (van den Heuvel-Panhuizen 2008, p. 243);

Culture

the totality of artefacts, rites, stories and customs shared in a given human social group (Ryan & Williams, 2007, p. 161).

Developmental progression

a sequence of levels of thinking (Clements and Sarama, 2007, p. 463).

Dynamic Geometric Software (DGS) programs

tools that can be used to construct and manipulate geometric objects and relations (e.g. Battista, 1998). They help children to develop rich mental models which help them to reason in increasingly sophisticated ways (Battista, 2001).

Embodied cognition

understanding situated in the body, in space and time, as well as socioculturally and historically (Ryan & Williams, 2007).

Embodiment

an idea or abstraction expressed or represented physically or concretely (Ryan & Williams, 2007). For example, young children can explore number operations on a floor number line, by moving themselves forward and back on the line. They often communicate and articulate their understandings and ideas by using actions and gestures instead of/as well as words.

Formal mathematical knowledge

knowledge that is school taught, largely represented in written form and frequently the result of deliberate efforts by children and teachers (Baroody et al., 2006, p. 189).

Hypothetical learning trajectory (HLT)

instructional sequences or potential developmental paths that serve to focus educators' attention on teaching children rather than on teaching a curriculum (Baroody et al., 2006, p. 206).

Informal mathematical knowledge

knowledge gleaned from everyday activities in what are not normally considered instructional settings such as home, playground, grocery store, family car. Such knowledge is usually represented verbally or nonverbally and often learned incidentally (Baroody et al., 2006, p. 189).

IRF

the teacher initiation–student response–teacher feedback (IRF sequence) is a form of classroom interaction commonly practiced in classroom discourse (Sinclair & Coulthard, 1975). The sequence is contrasted with a participation structure that allows for student-initiated negotiations.

Learning trajectory

description of children's thinking and learning of a specific mathematical domain and a conjectured route for that learning to follow through a set of instructional activities (Clements, 2008).

Mathematical model

a bridge between informal understanding and the abstraction of formal ideas. A model can for instance include materials, visual sketches or symbols. The models are formulated by children themselves in the course of their engagement with a problem (van den Heuvel-Panhuizen, 2003).

Mathematical processes

general mathematical processes such as problem-solving, reasoning and proof, communicating, connecting, and representing; justifying, argumentation; generalising;

mechanisms by which children can go back and forth between the abstract mathematics and real situations in the world around them (NRC, 2009, p. 43).

Mathematical proficiency

consists of the five intertwined and interrelated strands of conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition (NRC, 2001).

Mathematizing

the analysing of real-world problems in a mathematical way (Treffers & Beishuizen, 1999, p. 32);

casting children's actions (work) into explicitly mathematical form (e.g. Ginsburg 2009b, p. 123);

to conceive of problems in explicitly mathematical terms (Ginsburg 2009a, p. 412);

formulating real situations in mathematical terms (NRC, 2009, p. 43; 354);

involves reinventing, re-describing, reorganising, quantifying, structuring, abstracting, generalising, and refining that which is first understood on an intuitive and informal level in the context of everyday activity (Clements, Sarama & DiBiase, 2004, p. 12);

…organising information into charts and tables, noticing and exploring patterns, putting forth explanations and conjectures, and trying to convince one another of their thinking (Fosnot & Dolk, 2001, pp. 4–5);

…more than process is happening. Children [can be] exploring ideas such as quantity and unitizing, and division, in relation to their own level of mathematical development. And mathematizing should not be dismissed as simply process. Mathematizing is content. (Fosnot & Dolk, 2001, p.9)

Model context

can stand for a whole range of related arithmetic situations in which the operations of addition, subtraction, multiplication and division are meaningfully reflected. It can provide support in enabling children to carry out a calculation or develop a procedure (van den Heuvel-Panhuizen, 2008, p. 91).

Modeling problems

can be contrasted with traditional 'word' problems since the information given is often in a form (for example, a table) that must be interpreted by the child. Problems revolve around authentic situations that need to be interpreted and described in mathematical ways. (English & Watters, 2004)

Modelling

a process through which children learn how to behave as mathematicians by imitating (modelling) the behaviour of others. Adults can teach children how to act mathematically by presenting them with examples of the dispositions, attitudes and values which the adults around them consider to be appropriate. Modelling occurs when children internalise these behaviours
(Adapted from MacNaughton & Williams, 2004).

Procedural fluency

skill in carrying out procedures flexibly, accurately, efficiently and appropriately (NRC, 2001).

Productive disposition

the habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and in one's own efficacy (NRC, 2001).

Reflective abstraction

for Piagetians, reflective abstraction is a key process for activating accommodation and assimilation, or restructuring one's schema/models. This implies that children learn from talking about and reflecting on their mathematical ideas and solutions/strategies with others (Ryan & Williams, 2007, p. 158).

RME

is an acronym for Realistic Mathematics Education – an approach to mathematics education devised by Freudenthal in the Netherlands in the 1970s (see *Chapter 5*).

Routine expertise

mastery of basic skill and other skills by rote (Baroody et al., 2006, p. 2001).

Self-regulation

where the learner takes control and ownership of their own learning.

Specific language impairment (SLI)

a language disorder that delays the mastery of language skills in children who have no hearing loss or other developmental delays. SLI is also called developmental language disorder, language delay, or developmental dysphasia. It is one of the most common childhood learning disabilities, affecting approximately 7 to 8 percent of children in kindergarten. The impact of SLI persists into adulthood. Definition taken from *https://www.nidcd.nih.gov/health/voice/pages/specific-language-impairment.aspx*

Strategic competence

the ability to formulate, represent, and solve mathematical problems (NRC, 2001).

Working memory

relates to the task at hand, and coordinates the recall of memories necessary to complete it.

REFERENCES

References

Ackermann, E. (2001). Piaget's constructivism, Papert's constructionism: What's the difference? In *Constructivism: Uses and perspectives in education, Volume 1 & 2.* Conference proceedings, Geneva: Research Centre in Education, Cahier 8, September 01, pp. 85–94. Retrieved from http://learning.media.mit.edu/content/publications/EA.Piaget%20_%20Papert.pdf

Anderson, A., Anderson, J., & Shapiro, J. (2005). Supporting multiple literacies: Parents' and children's mathematical talk within storybook reading. *Mathematics Education Research Journal*, *16*(3), 5–26.

Anderson, A., Anderson, J., & Thauberger, C. (2008). Mathematics learning and teaching in the early years. In O. Saracho and B. Spodek (Eds.), *Contemporary perspectives on mathematics in early childhood education* (pp. 95–132). Charlotte, NC: Information Age.

Anderson, D., & Gold, E. (2006). Home to school: Numeracy practices and mathematical identities. *Mathematical Thinking and Learning. 8*(3), 261–286.

Anthony, G., & Walshaw, M. (2007). *Effective pedagogy in mathematics/Pangarau: Best Evidence Synthesis Iteration (BES).* New Zealand Ministry of Education.

Anderson, J., White, P., & Sullivan, P. (2005). Using a schematic model to represent influences on, and relationships between, teachers' problem-solving beliefs and practices. *Mathematics Education Research Journal, 17*(2), 9–38.

Australian Association of Mathematics Teachers/Early Childhood Australia (AAMT/ECA). (2006). *Position paper on early childhood mathematics.* Retrieved from http://www.aamt.edu.au/Publications-and-statements/Position-statements/Alternative-formats/Early-Childhood-Position-black

Baker, C. (2001). *Foundations of bilingual education and bilingualism* (3rd ed.). Clevedon: Multilingual Matters.

Barbarin O., Early, D., Clifford, R., Bryant, D., Frome, P., Burchinal, M., Howes, C., & Pianta, R. (2008). Parental conceptions of school readiness: Relation to ethnicity, socioeconomic status and children's skill's. *Early Education and Development*, *19*(5), 671–701.

Barnes, M. A., Fletcher, J. M., & Ewing-Cobbs, L. (2007). Mathematical disabilities in congenital and acquired neurodevelopmental disorders. In D. B. Berch, & M. M. Mazzocco (Eds.), *Why is math so hard for some children? The nature and origins of mathematical learning difficulties and disabilities.* Baltimore, MD: Paul H Brookes.

Baroody, A., Lai, M., & Mix, K. (2006). The development of young children's early number and operation sense and its implications for early childhood education. In B. Spodek & O. Saracho, (Eds.), *Handbook of research on the education of young children* (2nd ed.) (pp.187–221). Mahwah, NJ: Lawrence Erlbaum.

Baroody, A., Purpura, D., & Reid, E. (2012). Comments on learning and teaching early and elementary mathematics. In J. Carlson & J. Levin (Eds.), *Instructional strategies for improving students' learning: Focus on early reading and mathematics* (pp. 163–175). Charlotte, NC: Information Age.

Battista, M. (1998). *Shape makers: Developing geometric reasoning with the Geometer's Sketchpad*. Berkeley, CA: Key Curriculum Press.

Battista, M. (2001). Shapemakers. *Computers in the Schools*, *17*(1–2), 105–120.

Battista, M. (2004). Applying cognition-based assessment to elementary school students' development of understanding of area and volume measurement. *Mathematical Thinking and Learning*, *6(*2), 185–204.

Berch, D. B., &. Mazzocco, M. M. (Eds.), (2007). *Why is math so hard for some children? The nature and origins of mathematical learning difficulties and disabilities*. Baltimore: Paul H Brookes.

Berliner, D. (2011). *Why giving standardized tests to young children is 'Really dumb'*. US National Education Policy Centre. Retrieved from http://nepc.colorado.edu/blog/why-giving-standardized-tests-young-children-really-dumb

Bishop, A., & Forganz, H. (2007). Issues in access and equity in mathematics education. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 1145–1167). New York, NY: Information Age.

Bobis, J., Clarke, B., Clarke, D., Thomas, G., Wright, B., Young-Loveridge, J., & Gould, P. (2005). Supporting teachers in the development of young children's mathematical thinking: Three large scale cases. *Mathematics Education Research Journal, 16*(3), 27–57.

Bodrova, E., & Leong, D. (2007). *Tools of the mind: the Vygotskian approach to early childhood education.* Upper Saddle River, NJ: Pearson Education.

Bonotto, C. (2005). How informal out-of-school mathematics can help students make sense of formal in-school mathematics: The case of multiplying by decimal numbers. *Mathematical Thinking and Learning*, *7*(4), 313–344.

Brannon, E. M., Abbott, S., & Lutz, D. J. (2004). Number bias for the discrimination of large visual sets in infancy. *Cognition, 93,* B59-B68.

Bruner, J. (1966). *The growth of mind*. Newton, MA: American Psychological Society.

Bruner, J. (1973). *The relevance of education*. New York, NY: Norton.

Bruner, J. (1996). *The culture of education*. Cambridge, MA: Harvard University Press.

Bull, R. (2007). Neuropsychological factors. In D. B. Berch, & M. M. Mazzocco (Eds.), *Why is math so hard for some children? The nature and origins of mathematical learning difficulties and disabilities*. Baltimore: Paul H Brookes.

Burton, L. (2001). Research mathematicians as learners: And what mathematics education can learn from them. *British Educational Research Journal, 27*(5), 589–599.

Burton, L. (2004). *Mathematicians as enquirers: Learning about learning mathematics.* Boston and Dordrecht: Kluwer.

Butler, D. (2007). A constructionist view of what it means to be digitally literate. *The Nordic Journal of Digital Literacy, 2*, 61–77.

Butterworth, B. (2005). The development of arithmetical abilities. *Journal of Child Psychology and Psychiatry, 46*(1), 3–18.

Butterworth, B., & Reigiosa, V. (2007). Information processing deficits in dyscalculia. In D. B. Berch, & M. M. Mazzocco (Eds.), *Why is math so hard for some children? The nature and origins of mathematical learning difficulties and disabilities.* Baltimore: Paul H Brookes.

Cai, J., & Howson, G. (2013). Towards an international mathematics curriculum. In M. Clements, A. Bishop, C. Keitel, J. Fitzpatrick & S. Leung (Eds.), *Third international handbook of mathematics education* (pp. 949–974). Normal, IL: Springer.

Cannon, J., & Ginsburg, H. (2008). ''Doing the Math'': Maternal beliefs about early mathematics versus language learning. *Early Education and Development, 19*(2), 238–260.

Carey, D., Fennema, E., Carpenter, T. P., & Franke, M. L. (1995). Cognitively guided instruction: Towards equitable classrooms. In W. G. Secada, E. Fennema, & L. Byrd (Eds.), *New directions for equity in mathematics education* (pp. 93–125). Cambridge, England: Cambridge University Press.

Carey, S. (2004). Bootstrapping and the origins of concepts. *Daedalus, 133*, 59–68.

Carr, M. (2000). Let me count the ways (to do assessment). Proceedings of The *Practice, Policy and Politics Conference*, Victoria University of Wellington, 9–12 July 2000. New Zealand: New Zealand Educational Institute.

Carr. M. (2001). *Assessment in early childhood settings.* London, UK: Paul Chapman.

Carr, M., & Lee, W. (2012). *Learning stories: Constructing learner identities in early education.* London, UK: Sage.

Carruthers, E., & Worthington, M. (2006).*Children's mathematics: Making marks, making meaning.* London, UK: Sage.

Casey, B. (2009). Applying developmental approaches to learning math. In O. Barbarin and B. Wasik (Eds.), *Handbook of child development and early education: Research to practice* (pp. 478–498). New York, NY: The Guildford Press.

Castagnetti, M., & Vecchi, V. (1997). *Shoe and meter.* Reggio Emilia, Italy: Reggio children. Retrieved from http://www.reggiochildren.it/?libro=scarpa-e-metro&lang=en

CCSSM/NGA. (2010). *Common core state standards for mathematics.* Washington, DC: Council of Chief State School Officers and the National Governors' Association Centre for Best Practices. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Charalambous, Y., Delaney, S., Hsu, H-Y, & Mesa, V. (2010). A comparative analysis of the addition and subtraction of fractions in textbooks from three countries. *Mathematical Thinking and Learning*, *12*(2), 117–151.

Chin, N. B., & Wigglesworth, G. (2007). *Bilingualism: An advanced resource book*. Abingdon, UK: Routledge.

Chu, C., & Ju, C. (2010). Evolution of constructivism. *Contemporary Issues in Education*, *3*(4), 63–66.

Clarke, D. (2001). *Early numeracy research project (1999–2001): Summary of the final report*. Victoria: Department of Education, Employment and Training.

Clarkson, P. C. (1992). Language and mathematics: A comparison of bilingual and monolingual students in mathematics. *Educational Studies in Mathematics 23*(4), 417–29.

Clements, D. (2008). Linking research and curriculum development. Young children's access to powerful mathematical ideas. In L. English (Ed.), M. Bussi, G. Jones, R. Lesh, B. Sriraman, & D. Tirosh (Ass. Eds.), *Handbook of international research in mathematics education* (2nd ed.) (pp. 589–625). New York, NY: Routledge.

Clements, D., & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning*, *6*(2), 81–89.

Clements, D., & Sarama, J. (2007). Early childhood mathematics learning. In F. Lester (Ed.) *Second handbook of research on mathematics teaching and learning* (pp. 461–555). New York: Information Age.

Clements, D., & Sarama, J. (2008). Experimental evaluation of the effects of a research-based preschool mathematics curriculum. *American Educational Research Journal, 45*, 443–494.

Clements, D., & Sarama, J. (2009a). Learning trajectories in early mathematics –Sequences of acquisition and teaching. *Encyclopedia of Language and Literacy Development* (pp. 1–6). London, ON: Canadian Language and Literacy Research Network. Retrieved 7 Feb 2013 from http://literacyencyclopedia.ca/pdfs/topic.php?topId=270

Clements, D., & Sarama, J. (2009b). *Learning and teaching early math: The learning trajectories approach*. UK: Routledge.

Clements, D., & Sarama, J. (2012a). Learning and teaching early elementary maths. In J. Carlson, & J. Levin (Eds.), *Instructional strategies for improving students' learning: Focus on early reading and mathematics* (pp.107–162). Charlotte, NC: Information Age.

Clements, D., & Sarama, J. (2012b). Mathematics, learning, assessment and curriculum. In R. Pianta, (Ed.), W. Barnett, L. Justice, S. Sheridan (Ass. Eds.), *Handbook of early childhood education* (pp.217–239). New York, NY: The Guilford Press.

Clements, D., Sarama, J., & DiBiase, A. (Eds.), (2004). *Engaging young children in mathematics: Standards for early childhood mathematics education*. Mahwah, NJ: Lawrence Erlbaum.

Clements, D., Sarama, J., and Gerber, S. (2005, April). *Mathematics knowledge of entering preschoolers.* Paper presented at the Annual Meeting of the American Educational Research Association, Montreal, Canada.

Clements, M., Keitel, C., Bishop, A., Kilpatrick, J., & Leong, K. (2013). From the few to the many: Historical perspectives on who should learn mathematics. In M. Clements, A. Bishop, C. Keitel, J. Fitzpatrick & S. Leung (Eds.). *Third International handbook of mathematics education* (pp. 7–40). Normal, IL: Springer.

Close, S. (2006). The junior cycle curriculum and the PISA mathematics framework. *Irish Journal of Education, 37,* 53–78.

Cobb, P. (2007). Putting philosophy to work: Coping with multiple theoretical perspectives. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 3–38). New York, NY: Information Age.

Cobb, P., & Bowers, J. (1999). Cognitive and situated perspectives in theory and practice. *Educational Researcher, 28*(2), 4–15.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31*(3/4), 175–190.

Cobb, P., Yackel, E., & McClain, K. (Eds.), (2000). *Symbolizing and communicating in mathematics lessons; Perspectives on discourse, tools and instructional design*. Mahwah, NJ: Lawrence Erlbaum.

Cockcroft, W. H. (1982). *Mathematics counts: Report of the Committee of Inquiry into the teaching of mathematics in schools*. London, UK: HMSO.

Cole, M. (1996). *Cultural psychology: A once and future discipline*. Cambridge, MA: The Balknap Press of Harvard University Press.

Cole, M., & Engeström, Y. (1993). A cultural-historical approach to distributed cognition. In G. Salomon (Ed.), *Distributed cognitions: Psychological and educational consideration* (pp. 1–46). New York, NY: Cambridge University Press.

Collins, A., Brown, J. S., & Newman, S. E. (1990). Cognitive apprenticeship: Teaching the crafts of reading, writing, and mathematics. In L. B. Resnick (Ed.), *Knowing, learning, and instruction: Essays in honor of Robert Glaser* (pp. 453–494). Hillsdale, NJ: Lawrence Erlbaum.

Collins, A., Greeno, J., & Resnick, L. B. (1992). Cognition and learning. B. Berliner & R. Calfee (Eds), *Handbook of Educational Psychology.* New York, NY: Simon & Shuster MacMillan.

Commission of the European Communities, (2011). *Progress towards the common European objectives in education and training.* Brussels: Author.

Conway, P., & Sloane, F. (2005). *International trends in post primary mathematics education.* Dublin, Ireland: National Council for Curriculum and Assessment.

Copley, J. (1999). *Mathematics in the early years.* Reston, VA: NCTM/NAEYC.

Corcoran, D. (2008). *Developing mathematical knowledge for teaching: A three-tiered study of Irish pre-service primary teachers.* Unpublished doctoral thesis. University of Cambridge.

Corcoran, D. (2012). Constructivism made visible in contingency: Learning to teach mathematics in a community of practice. In F. Waldron, J. Smith, M. Fitzpatrick, & T. Dooley, *Re-imagining initial teacher education: Perspectives on transformations* (pp. 135–154). Dublin, Ireland: The Liffey Press.

Cosgrove, J., Shiel, G., Sofroniou, N., Zastrutiski, S., & Shortt, F. (2005). *Education for life: The achievements of 15-year olds in Ireland in the second cycle of PISA.* Dublin, Ireland: Educational Research Centre.

Crowley, M. L. (1987). The van Hiele model of the development of geometric thought. In M. M. Lindquist (Ed.), *Learning and teaching geometry K-12: 1987 yearbook of the National Council of Teachers of Mathematics* (pp. 1–16). Reston, VA: NCTM.

Cuban, L. (1993). *How teachers taught: Constancy and change in American classrooms, 1880–1990* (2nd ed.). New York, NY: Teachers College Press.

Cummins, J. (1976). The influence of bilingualism on cognitive growth: A synthesis of research findings and explanatory hypotheses. *Working Papers on Bilingualism (OISE), 9,* 1–43.

Cummins, J. (2000). *Language, power and pedagogy: Bilingual children in the crossfire.* Clevedon, UK: Multilingual Matters.

Daro, P., Mosher, F., & Corcoran, T. (2011). *Learning trajectories in mathematics: A foundation for standards, curriculum, assessment, and instruction.* US: Center for Continuous Instructional Improvement (CCII). Retrieved from http://www.ccii-cpre.org

Davis, A. (1991). The language of testing. In K. Durkin & B. Shire, *Language in mathematical education: Research and practice.* Milton Keynes, UK: Open University Press.

Davis, B., & Simmt, E. (2003). Understanding learning systems: Mathematics education and complexity science. *Journal for Research in Mathematics Education, 34(*2), 137–167.

Davis, P., & Florian, L. (2004). *Teaching strategies and approaches for pupils with special educational needs: A scoping study (Research Report 516).* Norwich, UK: Her Majesty's Stationery Office.

Davis, P., & Hersh, R. (1981). *The mathematical experience.* Boston, MA: Birkhäuser.

Department of Education/An Roinn Oideachas (1971). *Primary school curriculum (Vol. 1) /Curaclam na bunscoile (Iml. 1).* Dublin, Ireland: Brún agus Nualláin.

Department of Education and Science (DES), (2005). *An evaluation of curriculum implementation in primary schools. English, mathematics and visual arts.* Dublin, Ireland: Stationery Office.

Department of Education and Skills (DES), (2011). *Literacy and numeracy for learning and life: The national strategy to improve literacy and numeracy among children and young people 2011–2020.* Dublin, Ireland: Author.

Department of the Taoiseach (2008). *Building Ireland's smart economy: A framework for sustainable economic renewal.* Dublin, Ireland: Stationery Office.

Department of the Taoiseach (2011). *Programme for government.* Dublin, Ireland: Author.

Dickinson, D., & Tabors, P. (2001). *Beginning literacy with language: Young children learning at home and school.* Baltimore, MD: Paul H. Brookes.

Diezmann, C. M., Lowrie, T., Bicknell, B., Faragher, R., & Putt, I. (2004). Catering for exceptional children in mathematics. In B. Perry, G. Anthony, & C. M. Diezmann (Eds.), *Research in mathematics education in Australasia 2000–2003* (pp. 175–195). Flaxton, QLD: Post Pressed.

Diezmann, C. M., & Watters, J. J. (2002). Summing up the education of mathematically gifted students. *Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia* (pp. 219–226). Auckland, NZ: MERGA.

Donaldson, M. (1984). *Children's minds.* London, UK: Fontana.

Donlan, C. (2007). Mathematical development in children with specific language impairments. In D. B. Berch & M. M. Mazzocco (Eds.), *Why is math so hard for some children? The nature and origins of mathematical learning difficulties and disabilities.* Baltimore: Paul H Brookes.

Donlan, C. (2009). The role of language in mathematical development: Typical and atypical pathways. *Encyclopedia of language and literacy development* (pp. 1–7). London, ON: Canadian Language and Literacy Research Network. Retrieved from http://literacyencyclopedia.ca/pdfs/topic.php?topId=272

Donlan, C., Cowan, R., Newton, E. J., & Lloyd, D. (2007). The role of language in mathematical development: Evidence from children with specific language impairments. *Cognition, 103*, 23–33.

Dooley, T. (2011). Telling matters in mathematics teaching and learning. In T. Dooley, D. Corcoran, & M. Ryan (Eds.), *Proceedings of the fourth conference on research in mathematics education* (pp. 19–38). Dublin, Ireland: MEI.

Dooley, T., & Corcoran, D. (2007). Mathematics: A subject of rights and wrongs? In P. Downes & A. L. Gilligan (Eds.), *Beyond educational disadvantage* (pp. 216–228). Dublin, Ireland: IPA.

Dossey, J. A. (1992). The nature of mathematics: Its role and its influence. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 39–48). New York, NY: Macmillan.

Douglas, G., Travers, J., McLinden, M., Roberstson, C., Smith, E., Macnab, N., Powers, S., Guldberg, K., McGough, A., O Donnell, M., & Lacey, P. (2012). *Measuring educational engagement, progress and outcomes for children with special educational needs: A review.* Trim, Ireland: National Council for Special Education.

Dowker, N. (2004). *What works for children with mathematical difficulties?* Research report RR554. Nottingham, UK: DfES Publications.

Dreyfus, T., & Eisenberg, T. (1986). On the aesthetics of mathematical thought. *For the Learning of Mathematics, 6*(1), 2–10.

Drummond, M. J. (2012). *Assessing children's learning*. London, UK: Fulton.

Duffy, T. (2009). Building lines of communication and a research agenda. In S. Tobias & T. Duffy (Eds.), *Constructivist instruction: Success or failure* (pp. 351–36). New York, NY: Routledge.

Dunphy, E. (2008). *The framework for early learning: A background paper. Supporting early learning through formative assessment.* Retrieved from http://www.ncca.ie/uploadedfiles/Primary/EC_assessment_paper2.pdf

Dunphy, E. (2009). Early childhood mathematics teaching: Challenges, difficulties and priorities of teachers of young children in primary schools in Ireland. *International Journal of Early Years Education, 17*(1), 3–16.

Dunphy, E. (2012). Perspectives on early learning. In M. Mhic Mhathúna & M. Taylor (Eds.), *Early Childhood Education and Care* (pp. 203–213). Dublin, Ireland: Gill & Macmillan.

Dunphy, E. (2012). Children's participation rights in early childhood education and care: The case of early literacy and pedagogy. *International Journal of Early Years Education.*

Educational Research Centre. (2010). *Drumcondra tests of early numeracy.* Dublin, Ireland: Author.

Educational Research Centre. (2011). *Drumcondra tests of early numeracy. DTEN-S and DTEN-D. Manual.* Dublin, Ireland: Author.

Ehrlich, S. B., Levine, S. C., & Goldin-Meadow, S. (2006). The importance of gesture in children's spatial reasoning. *Developmental Psychology, 42*, 1259–1268.

Eivers, E., & Clerkin, A. (2012). *PIRLS and TIMSS 2011. Reading, mathematics and science outcomes for Ireland.* Dublin, Ireland: Educational Research Centre.

Eivers, E., Close, S., Shiel, G., Clerkin, A., Gilleece, L., & Kiniry, J. (2010). *The 2009 national assessments of mathematics and English.* Dublin, Ireland: Educational Research Centre.

Ellerton, N., Clarkson, P., & Clements, M. A. (2000). *Research in mathematics education in Australasia 1996–1999* (pp. 25–99). Sydney, Australia: MEGRA.

Empson, S. (2011). On the idea of learning trajectories: Promises and pitfalls. *The Mathematics Enthusiast*, *8*(3), 571–596.

Engerström, Y., Meittenen, R., & Punamaki, R. (Eds.), (1999). *Perspectives on activity theory.* Cambridge, UK: Cambridge University Press.

Erbas, A. K., & Aydogan Yenmez, A. (2011). The effect of inquiry-based explorations in a dynamic geometry environment on sixth grade students' achievements in polygons. *Computers & Education, 57*(4), 2462–2475.

Ernest, P. (1989). The impact of beliefs on the teaching of mathematics. In P. Ernest (Ed.), *Mathematics teaching: The state of the art* (Vol. 249–254, pp. 254). New York, NY: The Falmer Press.

Ernest, P. (1991). *The philosophy of mathematics education.* Briston, PA: The Falmer Press

Ernest, P. (2010). *Commentary 2 on reflections on theories of learning.* In B. Sriraman, & L. English (Eds.), *Theories of mathematics education: Seeking new frontiers* (pp. 53–61). Berlin, Germany: Springer Verlag.

Ernest, P., Greer, B., & Sriraman, B. (Eds.), (2009). *Critical issues in mathematics education.* Charlotte, NC: Information Age.

Farrell, M. P. (2011). Bilingual competence and students' achievement in physics and mathematics. *International Journal of Bilingual Education and Bilingualism*, 14*(3),* 335–345.

Fazio, B. B. (1994). The counting abilities of children with specific language impairment: A comparison of oral and gestural tasks. *Journal of Speech, Language and Hearing Research, 37,* 358.

Fazio, B. B. (1996). Mathematical abilities of children with specific language impairment: A 2-year follow-up. *Journal of Speech, Language and Hearing Research, 39,* 839–849.

Fazio, B. B. (1999). Arithmetic calculation, short-term memory, and language performance in children with specific language impairment: A 5-year follow-up. *Journal of Speech, Language and Hearing Research, 42,* 420–431.

Feigenson, L., & Carey, S. (2005). On the limits of infants' quantification of small object arrays. *Cognition, 97,* 295–313.

Fiore, B. (2012). *Assessment of young children: A collaborative approach.* New York, NY: Routledge.

Fisher, R. (1990). *Teaching children to learn.* Oxford, UK: Basil Blackwell.

Fleer, M., & Richardson, C. (2009). Cultural-historical assessment: Mapping the transformation of understanding. In A. Anning, J. Cullen, & M. Fleer (Eds.), *Early childhood education: Society and culture* (2nd ed.), (pp. 130–144). London, UK: Sage.

Fluck, M., Linnell, M., & Holgate, M. (2005). Does counting count for 3- to 4-year-olds? Parental assumptions about preschool children's understanding of counting and cardinality. *Social Development, 14,* 496–513.

Forsythe, S. (2007). Learning geometry through dynamic geometry software. *Mathematics teaching incorporating Micromath*, 202, 31–36.

Fosnot, C. T. (1996). Constructivism: A psychological theory of learning. In C. T. Fosnot (Ed.), *Constructivism: Theory, perspectives, and practice* (pp. 8–33). New York, NY: Teachers College Press.

Fosnot, C. T., & Dolk, M. (2001). *Young mathematicians at work: Constructing number sense, addition and subtraction.* Portsmouth, NH: Heinemann.

Fox, J. (2005/2006). Practical and theoretical perspectives of the Dutch learning-teaching trajectories. *Mathematics Teacher Education and Development, 7,* 90–96.

Freudenthal, H. (1971). Geometry between the devil and the deep sea. *Educational Studies in Mathematics, 3*(3), 413–435.

Freudenthal, H. (1973). *Mathematics as an educational task.* Dordrecht, Holland: D. Reidel.

Frobisher, L., Frobisher, A., Orton, A., & Orton, J. (2007). *Learning to teach shape and space: A handbook for students and teachers in the primary school.* Cheltenham, UK: Nelson Thornes.

Fuchs, L. S., Fuchs, D., Compton, D. L., Bryant, J. D., Hamlett, C. L., & Seethaler, P. M. (2007). Mathematics screening and progress monitoring at first grade: Implications for responsiveness to intervention. *Exceptional Children, 73*(3), 311–330.

Geary, D., Hoard, M. K., Nugent, L., & Byrd-Craven, J. (2007). Strategy use, long-term memory, and working memory capacity. In D. B. Berch, & M. M. Mazzocco (Eds.), *Why is math so hard for some children? The nature and origins of mathematical learning difficulties and disabilities.* Baltimore: Paul H Brookes.

Gelman, R., & Butterworth, B. (2005). Number and language: how are they related? *Trends in Cognitive Sciences, 9,* 6–10.

Gelman, R. K., & Gallistel, C. (1978). *The child's understanding of number.* Cambridge, MA: Harvard University Press.

Gentner, D. (2003). Why we're so smart. In D. Gentner and S. Goldin-Madow (Eds.), *Language in mind: Advances in the study of language and thought* (pp. 195–235). Cambridge: MA: MIT Press.

Gersten, R., Clarke, B., Jordan, N. C., Newman-Gonchar, R., Haymond, K., & Wilkins, C. (2012). Universal screening in mathematics for the primary grades: Beginnings of a research base. *Council for Exceptional Children, 78*(4), 423–445.

Gilleece, L., Shiel, G., Clerkin, A., & Millar, D. (2012). *The 2010 national assessments of English reading and mathematics in Irish-medium schools.* Dublin, Ireland: Educational Research Centre.

Ginsburg, H. (1997). *Entering the child's mind: The clinical Interview in psychological research and practice.* New York, NY: Cambridge University Press.

Ginsburg, H. (2009a). Early mathematical education and how to do it. In O. Barbarin and B. Wasik (Eds.), *Handbook of child development and early education: Research to practice* (pp. 403–428). New York, NY: The Guildford Press.

Ginsburg, H. (2009b). The challenge of formative assessment in mathematics education: children's minds, teachers' minds. *Human Development, 52*, 109–128.

Ginsburg, H., & Ertle, B. (2008). Knowing the mathematics in early childhood mathematics, In O. Saracho, & B. Spodek, *Contemporary perspectives on mathematics in early childhood education* (pp.45–66). Charlotte, NC: Information Age.

Ginsburg, H., Lee, J., & Boyd, J. (2008) Mathematics education for young children: What it is and how to promote it. *Social Policy Report*, Volume XX11, 1. Retrieved from http://www.scrd.org

Ginsburg, H., & Seo, K. (1999). The mathematics in children's thinking. *Mathematical Thinking and Learning, 1*(2), 113–129.

Government of Ireland (1999). *Primary school curriculum: Mathematics.* Dublin, Ireland: Stationery Office.

Government of Ireland (2012). *Junior Certificate mathematics syllabus. Foundation, higher and ordinary level. For examination in 2015 only.* Dublin: National Council for Curriculum and Assessment.

Government of South Australia (2001). *The South Australian Curriculum, Standards and Accountability Framework*. Perth, Australia: Department of Education, Training and Employment.

Gravemeijer, K. (2004). Local instruction theories as means of support for teachers in reform mathematics education. *Mathematical thinking and Learning, 6*(2), 105–128.

Greeno, J. G. (1991). Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education, 22*, 170–218.

Greeno, J. G. (1997). On claims that answer the wrong questions. *Educational Researcher, 26*(1), 5–17.

Gresalfi, M., & Lester, F. (2009). What's worth knowing in mathematics? In S. Tobias & T. Duffy (Eds.), *Constructivist instruction: Success or failure* (pp. 264–290). New York, NY: Routledge.

Griffin, S. (2004). Building number sense with Number Worlds: A mathematics programme for young children. *Early Childhood Research Quarterly, 19*, 173–180.

Griffin, S. (2005). Fostering the development of whole-number sense: Teaching mathematics in the primary grades. In M. S. Donovan, & J. Bransford (Eds.), National Research Council, *How students learn: Mathematics in the classroom. Committee on how people learn: A targeted report for teachers.* (pp. 257–308). Washington, DC: The National Academies Press.

Griffin, S., Case, R., & Siegler, R. (1994). Rightstart: Providing the central conceptual prerequisites for first formal learning of arithmetic to students at risk for school failure. In K. McGilly (Ed.), *Classroom learning: Integrating cognitive theory and classroom lessons* (pp.25–50). Boston, MA: MIT Press.

Gutiérrez, K. D., Sengupta-Irving, T., & Dieckmann, J. (2010). Developing a mathematical vision. Mathematics as a discursive and embodied practice. In J. N. Moschkovich (Ed.), *Language and mathematics education: Multiple perspectives and directions for research* (pp. 29–71). Charlotte, NC: Information Age.

Gutiérrez, R., & Dixon-Román, E. (2011). Beyond gap gazing: How can thinking about education comprehensively help us (Re) envision mathematics education? *Mapping equity and quality in mathematics education* (pp. 21–34). Berlin, Germany: Springer.

Hadamard, J. W. (1945). *Essay on the psychology of invention in the mathematical field.* Princeton, NJ: Princeton University Press.

Haliday, M. (1978). *Language as a social semiotic.* London, UK: Edward Arnold.

Handal, B., & Herrington, A. (2003). Mathematics teachers' beliefs and curriculum reform. *Mathematics Education Research Journal, 15*(1), 59–69.

Hannafin, R., Burruss, J., & Little, C. (2001). Learning with dynamic geometry programs: Perspectives of teachers and learners. *Journal of Educational Research*, *94*(3), 132–144.

Healy, L., & Hoyles, C. (2001). Software tools for geometrical problem solving: Potentials and pitfalls. *International Journal of Computers for Mathematical Learning, 6,* 235–256.

Herbel-Eisenmann, B. (2007). From intended curriculum to written curriculum: Examining the 'voice' of a mathematics textbook. *Journal for Research in Mathematics Education, 38*(4), 344–369.

Hersh, R. (1997). *What is mathematics, really?* Oxford, UK: Oxford University Press.

Hufferd-Ackles, K., Fuson, K., & Sherin, M. (2004). Describing levels and components of a math-talk learning community. *Journal for Research in Mathematics Education*, *35*(2), 81–116.

Hughes, M. (1986). *Children and number: Difficulties in learning mathematics.* Oxford, UK: Blackwell.

Hunting, R., Bobis, J., Doig, B., English, L., Mousley, J., Mulligan, J., Papic, M., Pearn, C., Perry, B., Robbins, J., & Young-Loveridge, J-Y. (2013). *Mathematical thinking of preschool children in rural and regional Australia: Research and practice.* Victoria, Australia: ACER Press.

Jaworski, B. (1992). Mathematics teaching: What is it? *For the Learning of Mathematics: An International Journal of Mathematics Education, 12(*1), 8–14.

Jeffes, J., Jones, E., Cunningham, R., Dawson, A., Cooper, L., Straw, S., Sturman, L., & O'Kane, M. (2012). *Research into the impact of Project Maths on student achievement, learning and motivation. First interim report.* Slough, UK: NFER.

Jeffes, J., Jones, E., Wilson, M., Lamont, E., Straw, S., Wheater, R., & Dawson, A. (2013). *Research into the impact of Project Maths on student achievement, learning and motivation.* Slough: NFER.

Jonassen, D. (1996). *Computers in the classroom: Mindtools for critical thinking.* Colombus, OH: Prentice Hall.

Jonassen, D., & Carr, C. (2000). Mindtools: Affording multiple knowledge representations for learning. In S. Lajoie, (Ed.), *Computers as cognitive tools: No more walls* (pp. 165–196). Mahwah, NJ: Lawrence Erlbaum.

Jonassen, D., & Peck. K. (1998). Learning WITH technology: A constructivist perspective. Colombus, OH: Prentice Hall.

Jonassen, D. H., Peck K. L., & Wilson, B. G. (1999). *Learning with technology: A constructivist perspective.* Upper Saddle River, NJ: Merrill/Prentice Hall.

Jonassen, D. & Reeves, T. (1996). Learning with technology: Using computers as cognitive tools. In D. Jonassen (Ed.), *Handbook of research for educational communications and technology* (pp. 693–719). New York, NY: MacMillan.

Jordan, N. C. (2007). Do words count? Connections between mathematics and reading difficulties. In D. B. Berch, & M. M. Mazzocco (Eds.), *Why is math so hard for some children? The nature and origins of mathematical learning difficulties and disabilities.* Baltimore, MD: Paul H Brookes.

Jordan, N. C. (2010). Early predictors of mathematics achievement and mathematics learning difficulties. In R. E. Tremblay, R. G. Barr, RdeV. Peters, M. Boivin (Eds.), *Encyclopedia on Early Childhood Development* [online]. Montreal, Quebec: Centre of Excellence for Early childhood Development, 1–6. Retrieved from http://www.child-encyclopedia.com/documents/JordanANGxp.pdf

Jordan, N. C., & Levine, S. C. (2009). Socioeconomic variation, number competence, and mathematics learning difficulties in young children. *Developmental Disabilities Research Reviews, 15*, 60–68.

Jordan, N. C., Kaplan, D., Locuniak, M. N., & Ramineni, C. (2007). Predicting first-grade math achievement from developmental number sense trajectories. *Learning Disabilities Research and Practice, 22(1),* 36–46.

Jordan, N. C., Kaplan, D., Oláh, L. N., & Locuniak, M. N. (2006). Number sense growth in kindergarten: A longitudinal investigation of children at risk for mathematics difficulties. *Child Development*, *77*, 153–175.

Kessler, C., & Quinn, M. (1982). Cognitive development in bilingual environments. In B. Hartford, A. Valdman, & C. Foster (Eds.) *Issues in international bilingual education: The role of the vernacular* (pp. 53–81). London, UK: Plenum.

Kirk, S., Gallagher, J. J., Coleman, M. R., & Anastasiow, N. (2012). *Educating exceptional children* (13th ed.). Belmont, CA: Wadsworth.

Kirschner, P. A., Sweller, J., & Clark, R. E. (2006). Why minimal guidance during instruction does not work: An analysis of the failure of constructivist, discovery, problem-based, experiential, and inquiry based teaching. *Educational Psychologist, 41,* 75–86.

Klibanoff, R., Levine, C., Huttenlocher, J., Hedges, L., & Vasilyeva, M. (2006). Preschool children's mathematical knowledge: The effect of teacher 'Math talk'. *Developmental Psychology*, *42*(1), 59–69.

Koestler, A. (1969). *The act of creation.* London, UK: Pan Books.

Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: interaction in classroom cultures.* Hillsdale, NJ: Lawrence Erlbaum.

Lakoff, G. & Nunez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being.* New York, NY: Basic Books.

Ladson-Billings, G. (1995). Making mathematics meaningful in multicultural contexts. In W. G. Secada, E. Fennema, & L. B. Adajian (Eds.), *New directions for equity in mathematics education* (pp. 126–145). Cambridge and New York: Cambridge University Press.

Lave, E. (1988). *Cognition in practice: Mind, mathematics and culture in everyday life.* Cambridge, UK: Cambridge University Press.

Lave, J. (1996). Teaching as learning in practice. *Mind, Culture and Activity*, *3(*3), 149–164.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation.* Cambridge, UK: Cambridge University Press.

Leder, G. C., Pehkonen, E., & Törner, G. (2003). *Beliefs: A hidden variable in mathematics education?* (Vol. 31). Dordrecht, The Netherlands: Kluwer.

Lerman, S. (1998). A moment in the zoom of a lens: Towards a discursive psychology of mathematics teaching and learning. In A. Olivier, & K. Newstead (Eds.). *Proceedings of the twenty-second annual meeting of the international group for the psychology of mathematics education,* Stellenbosch, South Africa. Volume 1, 66–81.

Lerman, S. (2000). Social turn in mathematics education research. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 19–44). Palo Alto, CA: Greenwood.

Levine, S. C., Ratliff, K. R., Huttenlocher, J., & Cannon, J. (2012). Early puzzle play: A predictor of preschoolers' spatial transformation skill. *Developmental Psychology, 48*(2), 530–542.

Lewis, C., & Tsuchida, I. (1998). A lesson is like a swiftly flowing river: How research lessons improve Japanese education. *American Educator, 12*, 14–17, 50–52.

Loewenstein, J., & Genter, D. (2005). Relational language and the development of relational mapping. *Cognitive Psychology, 50,* 315–353.

MacDonald, A. (2011). Young children's representations of their developing measurement understandings. In J. Clark, B. Kissane, J. Mousley, T. Spencer, & S. Thornton (Eds.), *Mathematics: Traditions and [new] practices* (Proceedings of the 34th annual conference of the Mathematics Education Research Group of Australia) and the 23rd Biennual conference of the Australian Association of Mathematics Teachers, Alice Springs, (pp. 482–490). Adelaide, Australia: AAMT & MERGA.

MacDonald, A. (2012). Young children's photographs of measurement in the home. *Early Years, 32*(1), 71–85.

MacDonald, A., Davies, N., Dockett, S., & Perry, B. (2012). Early childhood mathematics education. In B. Perry, T. Lowrie, T. Logan, A. MacDonald, & J. Greenlees (Eds.), *Research in mathematics education in Australia* 2008–2011 (pp. 169–192). Rotterdam, The Netherlands: Sense.

MacDonald, M. (2007). Towards formative assessment: the use of pedagogical documentation in early elementary classrooms. *Early Childhood Research Quarterly, 22*, 232–242.

MacNaughton, G., & Williams, G. (2004). (2nd ed.) *Techniques for teaching young children.* Frenchs Forest, Australia: Pearson.

MacSwan, J. (2000). The threshold hypothesis, semi-lingualism and other contributions to a deficit view of linguistic minorities. *Hispanic Journal of Behavioral Science, 22(1),* 3–45.

Malloy, C. E. (1999). Developing mathematical reasoning in the middle grades: Recognizing diversity. In L. V. Stiff, & F. R. Curcio (Eds.), *Developing mathematical reasoning in grades K–12: 1999 yearbook* (pp. 13–21). Reston, VA: NCTM.

Marschark, M., & Spencer, P. E. (2009). *The education of deaf and hard-of-hearing children: An international review.* Research Report 1. Trim, Ireland: National Council for Special Education.

Mazzocco, M. (2009). Cognitive processes underlying numeracy. *Encyclopedia of Language and Literacy Development* (pp. 1–11). London, ON: Canadian Language and Literacy Research Network. Retrieved from http://www.literacyencyclopedia.ca/pdfs/topic.php?topId=285

Mazzocco, M., Murphy, M. M., & McCloskey, M. (2007). The contribution of syndrome research to understanding mathematical learning disability: The case of Fragile X and Turner syndromes. In D. B. Berch, & M. M. Mazzocco (Eds.), *Why is math so hard for some children? The nature and origins of mathematical learning difficulties and disabilities*. Baltimore: Paul H Brookes.

McKenzie, J. (1998). Strategic teaching in concert with new technologies. *From now on. The Educational Technology Journal*, *8*(4). Retrieved from http://www.fno.org/dec98/strategic.html

Meira, L. (1995). The microevolution of mathematical representations in children's activity. *Cognition & Instruction. 13,* 269–313.

Mhic Mhathúna, M. (1999). Early steps in bilingualism: Learning Irish in Irish language immersion pre-schools. *Early Years: An International Research Journal*, *19(2),* 38–50.

Mueller, M. (2011). Math assessment in middle childhood. *Encyclopedia of language and literacy development* (pp. 1–9). London, ON: Western University. Retrieved from http://www.literacyencyclopedia.ca/pdfs/topic.php?topId=304

Mullis, I. V. S., Martin, M. A., Foy, P., & Arora, A. (2012). *TIMSS 2011 international results in mathematics.* Chestnut Hill, MA: Boston College: TIMSS and PIRLS International Study Centre.

Murphy, D., & Travers, J. (2012). Including young bilingual learners in the assessment process: A study of appropriate early literacy assessments utilising both languages of children in a Gaelscoil. In T. Day, & J. Travers (Eds.), *Special and inclusive education: A research perspective.* Oxford, UK: Peter Lang.

Nardi, E., & Steward, S. (2003). Is Mathematics T. I.R. E.D? A profile of quiet disaffection in the secondary mathematics classroom. *British Educational Research Journal, 29*(3), 345–367.

National Association for the Education of Young Children & National Council of Teachers of Mathematics (NAEYC/NCTM) (2002/2010). *Position statement: Early childhood mathematics: Promoting good beginnings.* Retrieved from http://www.naeyc.org/files/naeyc/file/positions/ProfPrepStandards09.pdf

National Council for Curriculum and Assessment (NCCA) (2005). *Primary curriculum review. Phase 1.* Dublin, Ireland: Author.

National Council for Curriculum and Assessment (NCCA) (2007). *Assessment in the primary school curriculum: Guidelines for schools.* Dublin, Ireland: Author.

National Council for Curriculum and Assessment (NCCA) (2009a). *Aistear: the Early Childhood Curriculum Framework. Principles and Themes.* Dublin, Ireland: Author.

National Council for Curriculum and Assessment (NCCA) (2009b). *Aistear: the Early Childhood Curriculum Framework. Guidelines for Good Practice.* Dublin, Ireland: Author.

National Council of Teachers of Mathematics (NCTM) (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics (NCTM) (2006). *Curriculum focal points for prekindergarten through grade 8 mathematics: a quest for coherence.* Reston, Virginia: NCTM. Retrieved from http://www.nctmmedia.org/cfp/front_matter.pdf

National Curriculum Board (NCB) (2009). *National mathematics curriculum: Framing paper.* Melbourne, Australia: Author.

National Research Council (NRC) (2001). *Adding it up: Helping children to learn mathematics.* J. Kilpatrick, J. Swafford, & B. Findell (Eds.), Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: The National Academies Press.

National Research Council (NRC) (2005). *How students learn: Mathematics in the classroom.* Committee on How people Learn: A Targeted Report for Teachers, M. S. Donovan, & J. Bransford (Eds.), Division of Behavioral and Social Sciences and Education. Washington, DC: The National Academies Press.

National Research Council (NRC) (2009). *Mathematics learning in early childhood: Paths towards excellence and equity.* Committee on Early Childhood Mathematics, C. Cross, T. Woods, and H. Schweingruber (Eds.), Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: The National Academies Press.

Neuman, S. B., Newman, E. H., & Dwyer, J. (2011). Educational effects of a vocabulary intervention on pre-schoolers' word knowledge and conceptual development: A cluster, randomized trial. *Reading Research Quarterly, 46*(3), 249–272.

NicMhuirí, S. (2011). Look who's talking now – the role of talk in relation to mathematical thinking and problem solving. In T. Dooley, D. Corcoran, and M. Ryan (Eds.), *Proceedings of the Fourth Conference on Research in Mathematics Education (MEI 4)*, (pp. 320–332). Dublin, Ireland: MEI.

Ní Ríordáin, M., & O'Donoghue, J. (2007). Language, mathematics meaning and Gaelgeoirí: A study of educational transitions in Irish education. In S. Close, D. Corcoran, and T. Dooley (Eds.), *Walking the Talk-Using Mathematics Education Research: Proceedings of the Second National Conference on Research in Mathematics Education* (168–182). Dublin, Ireland: MEI.

Norman, K. (Ed.) (1993). *The national oracy project.* London, UK: Hodder and Stoughton.

Nunes, T. (2004). *Teaching mathematics to deaf children.* London, UK: Whurr.

Nunes, T., Carraher, D. W., & Schliemann, A. D. (1993). *Street mathematics and school mathematics.* Cambridge, UK: Cambridge University Press.

Nyaumwe, L. (2004). The impact of full-time student teaching on preservice teachers' conceptions of mathematics teaching and learning. *Mathematics Teacher Education and Development, 6*, 19–30.

OECD (2010). *PISA 2009 results: What students know and can do – Student performance in reading, mathematics and science (Volume I).* Paris, France: Author.

O'Halloran, K. L. (2005). *Mathematical discourse: Language, symbolism and visual images.* New York, NY: Continuum.

Olkun, S., Sinoplu, N., & Deryakulu, D. (2005). Geometric explorations with dynamic geometry applications based on van Hiele levels. *International Journal for Mathematics Teaching and Learning*, 1–12.

Papert, S. (1980). *Mindstorms: Children, computers and powerful ideas.* New York, NY: Basic.

Papert, S. (1993). *The children's machine: Rethinking schools in the age of the computer.* New York, NY: Basic.

Papert, S., & Harel, I. (1991). Situating constructionism. In S. Papert & I. Harel (Eds.), *Constructionism.* New York, NY: Ablex. Retrieved from http://www.papert.org/articles/SituatingConstructionism.html

Peal, E., & Lambert. W. (1962). The relationship of bilingualism to intelligence. *Psychological Monographs 76*(27), 1–23.

Pengelly, H. (1991). *Triangles.* Melbourne, Australia: Ashton Scholastic.

Perkins, R., Cosgrove, J., Moran, G., & Shiel, G. (2012). PISA 2009: *Results for Ireland and changes since 2009.* Dublin, Ireland: Educational Research Centre.

Perkins, R., Shiel, G., Merriman, B., Cosgrave, J., & Moran, G. (2013). *Learning for life: The achievements of 15-year-olds in Ireland on mathematics, reading literacy and science in PISA 2012.* Dublin: Educational Research Centre.

Perry, B., & Dockett, S. (2002). Young children's access to powerful mathematical ideas. In L. D. English (Ed.), *Handbook of international research in mathematics education: Directions for the 21st century* (pp. 81–111). Mahwah, NJ: Laurence Erlbaum.

Perry, B., & Dockett, S. (2008). Young children's access to powerful mathematical ideas. In L. English (Ed.), M. Bussi, G. Jones, R. Lesh, B. Sriraman, & D. Tirosh (Ass. Eds.), *Handbook of international research in mathematics education* (2nd ed.) (pp. 75–108). New York, NY: Routledge.

Perry, B., Dockett, S., & Harley, E. (2007). Learning stories and children's powerful mathematics. *Early Childhood Research and Practice*, *9*(2), 1–14.

Piaget, J. (1952). *The origins of intelligence in children.* New York, NY: International Universities Press.

Pimm, D. (1991). Communicating mathematically. In K. Durkin & B. Shire (Eds.), *Language in mathematical education: research and practice* (pp.17–33). Bristol, PA: Open University Press.

Plowman, L, Stephen, C., & McPake, J. (2010). *Growing up with technology: Young children learning in a digital world.* London, UK: Routledge.

Porter, J. (1998). The understanding of counting in children with severe learning difficulties and nursery children. *British Journal of Educational Psychology, 68*, 331–45.

Porter, J. (1999). The attainments of pupils with severe learning difficulties on a simple counting and error detection task. *Journal of Applied Research in Intellectual Disabilities, 12*(2), 87–99.

Pound, L. (1999). *Supporting mathematical development in the early years.* Milton Keynes, UK: Open University Press.

Povey, H. (2002). Promoting social justice in and through the mathematics curriculum: Exploring the connections with the epistemologies of mathematics. *Mathematics Education Research Journal, 14*(3), 190–201.

Reeves, T. (1998). The impact of media and technology in schools a research report prepared for The Bertelsmann Foundation. Retrieved from *http://treeves.coe.uga.edu/edit6900/BertelsmannReeves98.pdf*

Remillard, J. T. (2005). Key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research, 75*(2), 211–246.

Resnick, L. (1987). *Education and learning to think.* Committee on Mathematics, Science and Technology Education; Committee on Behavioural and Social Sciences and Education, National Research Council. Washington, DC: National Acadamy Press.

Reynolds, R., & Harel-Caperton, I. (2011). Contrasts in student engagement, meaning-making, dislikes, and challenges in a discovery-based program of game design learning. *Education Technology Research Development, 59*, 267–289.

Rips, L. J., Asmuth, J., & Bloomfield, A. (2008). Do children learn the integers by induction? *Cognition, 106*, 940–951.

Robitaille, D. F., & Garden, R. A. (1989). *The IEA study of mathematics II: Content and outcomes of school mathematics.* Oxford, UK: Pergamon Press.

Rogers, W. T., Lin, J., & Rinaldi, C. M. (2011). Validity of the simultaneous approach to the development of equivalent achievement tests in English and French. *Applied Measurement in Education, 24*, 39–70.

Rogoff, B. (1998). Cognition as a collaborative process, In W. Damon (Ed.) *Handbook of child psychology.* (5th ed.). New York, NY: John Wylie.

Ross, J. A., McDougall, D., Hogaboam-Gray, A., & LeSage, A. (2003). A survey measuring elementary teachers' implementation of standards-based mathematics teaching. *Journal for Research in Mathematics Education, 34*(4), 344–363.

Roth, W-M., & Lee, Y-J. (2007). "Vygotsky's neglected legacy": Cultural-historical activity theory. *Review of Educational Research, 77*(2), 186–232. DOI: 10.3102/0034654306298273.

Rowland, T., Turner, F., Thwaites, A., & Huckstep, P. (2009). *Developing primary mathematics teaching.* London, UK: Sage.

Ryan, J., & Williams, J. (2007). *Children's mathematics 4–15: Learning from errors and misconceptions.* Maidenhead, UK: Open University Press.

Salomon, G. (1993). No distribution without individuals' cognition: a dynamic interactional view. In G. Salomon, (Ed.), *Distributed cognitions: Psychological and educational considerations* (pp. 111–4, 128–35). Cambridge, UK: Cambridge University Press.

Saracho, O., & Spodek, B. (2008). *Contemporary perspectives on mathematics in early childhood education.* Charlotte, NC: Information Age.

Sarama, J., & Clements, D. (2004). Building blocks for early childhood mathematics. *Early Childhood Research Quarterly, 19,* 181–189.

Sarama, J., & Clements, D. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. New York and London: Routledge.

Sarnecka, B. W., & Carey, S. (2008). How counting represents number: What children must learn and when they learn it. *Cognition, 108*, 662–674.

Saxe, G. B., Guberman, S. R., & Gearhart, M. (1987). Social processes in early number development. *Monographs of the Society for Research in Child Development, 52*(2).

Schleppegrell, M. J. (2010). Language in mathematics teaching and learning: A research review. In J. N. Moschkovich (Ed.), *Language and mathematics education: Multiple perspectives and directions for research* (pp. 73–112). Charlotte, NC: Information Age.

Schoenfeld, A. (2002). Making mathematics work for all children: Issues of standards, testing and equity. *Educational Researcher, 31*(1), 13–25.

Secada, W. G. (1995). Social and critical dimensions for equity in mathematics education. In W. G. Secada, E. Fennema, & L. B. Adajian (Eds.), *New directions for equity in mathematics education* (pp. 146–164). Cambridge and New York: Cambridge University Press.

Sfard, A. (2001). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics, 46*(1), 13–57.

Sfard, A. (2007). When the rules of discourse change, but nobody tells you: Making sense of mathematics learning from a commognitive standpoint. *Journal of Learning Sciences, 16*(4), 567–615.

Shallcross, W. L., Göksun, T., Golinkoff, R. M., Hirsh-Pasek, K., Lloyd, M., Newcombe, N., & Roseberry, S. (2008, March). *Building talk: Parental utterances during construction play.* Poster presented at the 16th International Conference on Infant Studies, Vancouver, Canada.

Shiel, G., Cregan, A., McGough, A., & Archer, P. (2012). *Oral language in early childhood and primary education (3–8 years).* Dublin, Ireland: National Council for Curriculum and Assessment.

Siegler, R. S. (2007). Foreword: The birth of a new discipline. In D. B. Berch, & M. M. Mazzocco (Eds.), *Why is math so hard for some children? The nature and origins of mathematical learning difficulties and disabilities.* Baltimore, MD: Paul H Brookes.

Sierra, J. (2008). Assessment of bilingual education in the Basque country. *Language, Culture and Curriculum, 21*(1), 39–47.

Silver, E. A., & Smith, M. S. (1996). Building discourse communities in mathematics classrooms: A worthwhile but challenging journey. In P. Elliott and M. Kenney (Eds.), *1996 Yearbook – Communication in mathematics, K-12 and beyond.* Reston, VA: NCTM.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114–145.

Simon, M. (2004) Key developmental understandings in mathematics: A direction for investigating and establishing learning goals. *Mathematical Thinking and Learning, 8*(4), 359–371.

Simon, M. (2006). Key developmental understandings in mathematics: A direction for investigating and establishing learning goals. *Mathematical Thinking and Learning, 8*(4), 359–371.

Simon, M., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning, 6*(2), 91–104.

Siraj-Blatchford, I., Sylva, K., Muttock, S., Gilden, R., & Bell, D. (2002). *Researching effective pedagogy in the early years (REPEY): DFES Research Report 356.*London, UK: DFES, HMSO.

Smith-Chant, B. (2010a). Learning trajectories in early mathematics-sequences of acquisition and teaching: A commentary on Clements and Sarama. *Encyclopedia of Language and Literacy Development* (pp. 1–4). London, ON: Western University. Retrieved from http://www.literacyencyclopedia.ca/pdfs/topic.php?topId=288

Smith-Chant, B. (2010b). Methods of assessment of young children's informal mathematical experiences: A commentary on Tudge. *Encyclopedia of Language and Literacy Development* (pp. 1–4). London, ON: Western University. Retrieved from http://www.literacyencyclopedia.ca/pdfs/topic.php?topId=289

Snow, C. E., & Van Hemel, S. (2008). *Early childhood assessment: Why, what, how.* Washington, DC: National Academies Press.

Sohmer, R., Michaels, S., O'Connor, M. C., & Resnick, L. (2009). Guided construction of knowledge in the classroom: The troika of talk, tasks and tools. In B. Schwartz, T. Dreyfus, & R. Hershkowitz (Eds.), *Transformation of knowledge through classroom* interaction (pp. 105–129). London and New York: Routledge.

Sophian, C. (2004). A prospective developmental perspective on early mathematics instruction. In D. Clements, J. Sarama, & A. DiBiase (Eds.), *Engaging young children in mathematics: Findings of the 2000 national conference on standards for preschool and kindergarten mathematics education (pp. 253–266).* Hillsdale, NJ: Lawrence Erlbaum.

Sophian, C. (2008). Rethinking the starting point for mathematics learning. In O. Saracho, & B. Spodek (Eds.), *Contemporary perspectives on mathematics in early childhood education* (pp. 21–44). Charlotte, NC: Information Age.

Sriraman, B., & English, L. (Eds.) (2010a). *Theories of mathematics education: Seeking New Frontiers.* Berlin, Germany: Springer Verlag.

Sriraman, B., & Haverhals, N. (2010). Lakatos-Hersh -Ernest: Triangulating Philosophy-Mathematics-Mathematics Education. In B. Sriraman, & L. English (Eds.), *Theories of mathematics education: Seeking new frontiers* (pp. 35–38). Berlin, Germany: Springer Verlag.

Sriraman, B. & Nardi, E. (2013). Theories in mathematics education: Some developments and ways forward. In M. Clements, A. Bishop, C. Keitel, J. Fitzpatrick, & S. Leung (Eds.). *Third International handbook of mathematics education* (pp. 303–325). Normal, IL: Springer.

Steffe, L. (1992). Building a foundation. *Journal for Research in Mathematics Education*, *23*(2), 182–186.

Steffe, L. (2004). On the construction of learning trajectories of children: The case of commensurate fractions. *Mathematical Thinking and Learning, 6*, 129–162.

Steffe, L. P., & Cobb, P. (1988). Construction of arithmetical meanings and strategies. New York, NY: Springer-Verlag.

Steffe, L. & Kieren, T. (1994). Radical constructivism and mathematics education. *Journal for Research in Mathematics Education, 25*, 711–733.

Stemhagen, K. (2009). Social justice and mathematics: Rethinking the nature and purposes of school mathematics. In P. Ernest, B. Greer, & B. Sriraman (Eds.), *Critical issues in mathematics education* (pp. 337–349). Charlotte, NC: Information Age.

Stewart, I. (1996). *Nature's numbers: Discovering order and pattern in the universe.* London, UK: Phoenix.

Stigler, J., & Thompson, B. (2012). You can't play twenty questions with mathematics teaching and learning, and win. In J. Carlson & J. Levin (Eds.), *Instructional strategies for improving students' learning: Focus on early reading and mathematics* (pp. 187–195). Charlotte, NC: Information Age.

Struik, D. J. (1987). *A concise history of mathematics* (4th ed.). New York, NY: Dover.

Swanson, H. L. (2007). Cognitive aspects of math disabilities. In D. B. Berch, & M. M. Mazzocco (Eds.), *Why is math so hard for some children? The nature and origins of mathematical learning difficulties and disabilities*. Baltimore, MD: Paul H Brookes.

Sweller, J. (2009). What human cognitive architecture tells us about construction. In S. Tobias, & T. Duffy (Eds.) (2009). *Constructivist instruction: Success or failure* (pp. 127–143). New York, NY: Routledge.

Sylva, K., Melhuish, E. C., Sammons, P., Siraj-Blatchford, I., & Taggart, B. (2004), *The Effective Provision of Pre-School Education (EPPE) Project: Technical Paper 12 – The final report: Effective pre-school education.* London, UK: DfES/Institute of Education, University of London.

Szechter, L. E., & Liben, L. (2004). Parental guidance in preschoolers' understanding of spatial-graphic representation. *Child Development, 5*, 869–885.

Tabors, P. O. (2008). *One child, two languages: A guide for early childhood educators of children learning English as a second language* (2nd ed). Baltimore, MD: Paul Brookes.

Thompson, A. G. (1984). The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice. *Educational Studies in Mathematics, 15*(2), 105–127.

Törner, G., Rolka, K., Rösken, B., & Sriraman, B. (2010). Understanding a teacher's actions in the classroom by applying Schoenfeld's theory 'Teaching-In-Context': Reflecting on goals and beliefs. In L. English & B. Sriraman (Eds.), *Theories of mathematics education: Seeking new frontiers* (pp. 401–420). Berlin, Germany: Springer.

Treffers, A. (1987). *Three dimensions: A model of goal and theory description in mathematics instruction – The Wiskobas project*. Dordrecht, The Netherlands: Reidel.

Treffers, A., & Beishuizen, M. (1999). Realistic mathematics education in the Netherlands. In I. Thompson (Ed.), *Issues in teaching numeracy in primary schools* (pp. 27–38). Milton Keynes, UK: Open University Press.

Turkle, S. (1995). Computers as evocative objects. From projective screens to relational objects. In J. Rassens, and J. Goldstein (Eds), *Handbook of computer games studies* (pp. 267–279). Cambridge, MA: MIT Press.

Valero, P. (2009). What has power got to do with mathematics education? In P. Ernest, B. Greer, & B. Sriraman (Eds.), *Critical issues in mathematics education* (pp. 237–254). Charlotte, NC: Information Age.

Van de Walle, J., Karp, K., & Bay-Williams, J. (2010). *Elementary and middle school mathematics.* Upper Saddle River, NJ: Pearson.

van den Heuvel-Panhuizen, M. (2003). The didactical use of models in Realistic Mathematics Education: An example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics*, *54*, 9–35.

van den Heuvel-Panhuizen, M. (Ed.) (2008). *Children learn mathematics: A learning-teaching trajectory: With intermediate attainment targets for calculations with whole numbers in primary schools.* Rotterdam, The Netherlands: Sense.

van den Heuvel-Panhuizen, M., & Buys, K. (Eds.) (2008). *Young children learn measurement and geometry. A learning-teaching trajectory with intermediate attainment targets for the lower grades in primary school.* Rotterdam, The Netherlands: Sense.

Von Glasersfeld, E. (1984). An introduction to radical constructivism. In P. Watzlawick (Ed.) *The invented reality* (pp. 17–40). New York, NY: Norton.

Von Glasersfeld, E. (1989). Cognition, construction of knowledge and teaching. *Synthese*, *801*, 121–140. New York, NY: Norton.

Vygotsky, L. (1978). *Mind in society.* Cambridge, MA: Harvard University Press.

Wager, A., & Carpenter, T. (2012). Learning trajectories through a sociocultural lens. In J. Carlson & J. Levin (Eds.), *Instructional strategies for improving students' learning: Focus on reading and mathematics* (pp. 197–204). Charlotte, NC: Information Age.

Wenger, E. (1998). *Communities of Practice*. Cambridge, UK: Cambridge University Press.

Wheeler, R. E., & Wheeler, E. R. (1979). *Mathematics: An everyday language* (4th ed.). New York, NY: John Wiley.

Whitin D. J., & Whitin, P. (2003). Talk counts. Discussing graphs with young children. *Teaching Children Mathematics*, *10*, 142–149.

Wiliam, D. (2007). Keeping learning on track: Classroom assessment and the regulation of learning. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 1053–1098). New York, NY: Information Age.

Wood, A., & Coltman, P. (1998). Talking mathematics. In E. Bearne (Ed.) *Use of language across the primary curriculum* (pp. 113–126). London, UK: Routledge.

Wright, R. J. (1994). A study of the numerical development of 5-year-olds and 6-year olds, *Educational Studies in Mathematics*, *26*, 25–44.

Wright, R., Martland, J., & Stafford, A. (2006). *Early numeracy: Assessment for teaching and intervention* (2nd ed.). London, UK: Sage.

Xu, F., Spelke, E. S., & Goddard, S. (2005). Number sense in human infants. *Developmental Science, 8*(1), 88–101.

Yelland, N., & Kilderry, A. (2010). Becoming numerate with information and communications technologies in the twenty-first century. *International Journal of Early Years Education*, *18*(2), 91–106.

Young-Loveridge, J. (2011). Assessing the mathematical thinking of young children in New Zealand: The initial school years. *Early Child Development and Care*, *181*(2), 267–276.

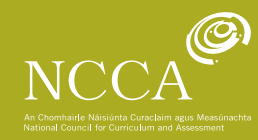
Zamarian, López-Rolón, A., & Delazer, M. (2007). Neuropsychological case studies on arithmetic processing. In D. B. Berch, & M. M. Mazzocco (Eds.), *Why is math so hard for some children? The nature and origins of mathematical learning difficulties and disabilities.* Baltimore, MD: Paul H. Brookes.

Zentall, S. (2007). Math performance of students with ADHD: Cognitive and behavioural contributors and interventions. In D. B. Berch, & M. M. Mazzocco (Eds.), *Why is math so hard for some children? The nature and origins of mathematical learning difficulties and disabilities.* Baltimore: Paul H Brookes.

Zevenbergen, R., Dole, S., & Wright, R. J. (2004). *Teaching mathematics in primary school.* Crows Nest, Australia: Allen and Unwin.

Zhou, X., Wang, Y., Wang, L., & Wang, B. (2006). Kindergarten children's representation and understanding of written number symbols. *Early Child Development and Care*, *176*(1), 33–45.

Zur, O., & Gelman, R. (2004). Young children can add and subtract by predicting and checking. *Early Childhood Research Quarterly, 19,* 121–137.


NCCA
An Chomhairle Náisiúnta Curaclaim agus Measúnachta
National Council for Curriculum and Assessment

NCCA

An Chomhairle Náisiúnta Curaclaim agus Measúnachta
National Council for Curriculum and Assessment

© NCCA 2014
ISSN 1649-3362

National Council for Curriculum and Assessment
35 Fitzwilliam Square, Dublin 2.

T: +353 1 661 7177
F: +353 1 661 7180
E: info@ncca.ie

www.ncca.ie

Design by www.principle.ie