

Sasa-Satsuma (complex modified Korteweg–de Vries II) and the complex sine-Gordon II equation revisited: Recursion operators, nonlocal symmetries, and more

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We present a new symplectic structure and a hereditary recursion operator for the Sasa-Satsuma equation which is widely used in nonlinear optics. Using an integrodifferential substitution relating this equation to a third-order symmetry flow of the complex sine-Gordon II equation enabled us to find a hereditary recursion operator and higher Hamiltonian structures for the latter equation. We also show that both the Sasa-Satsuma equation and the third-order symmetry flow for the complex sine-Gordon II equation are bi-Hamiltonian systems, and we construct several hierarchies of local and nonlocal symmetries for these systems. © 2007 American Institute of Physics. [DOI: 10.1063/1.2710552]

I. INTRODUCTION

Finding a recursion operator for a system of partial differential equations (PDEs) is of paramount importance, as the whole integrable hierarchy for the system in question is then readily generated by the repeated application of the recursion operator to a suitably chosen seed symmetry (see, e.g., Refs. 7, 32, and 11 and references therein). Moreover, using formal adjoint of the recursion operator enables us to produce infinitely many conserved quantities for our hierarchy (see, e.g., Refs. 7 and 11 for further details).

The recursion operators of multicomponent systems often have a richer structure of nonlocalities than in the one-component case, which makes such operators more difficult to find. For instance, there are two-component integrable generalizations [Eqs. (2) and (3)] of the sine-Gordon equation and of the modified Korteweg–de Vries (mKdV) equation, respectively. No recursion operator was found for Eqs. (2) and (3) so far even though the corresponding Lax pairs^{36,17,38} and bilinear representations^{17,20,18} are well known. The goal of the present paper is to fill this gap. Namely, we find and study the recursion operators for the so-called complex sine-Gordon II equation [Eq. (2)] and for the Sasa-Satsuma equation [Eq. (3)].

The Sasa-Satsuma equation³⁶ (see also Refs. 46 and 44) for a complex function U has the form

$$U_t = U_{xxx} + 6U\bar{U}U_x + 3U(U\bar{U})_x. \quad (1)$$

Here and below, the bar refers to the complex conjugate.

It is natural to refer to this equation as to the *complex mKdV II* as Eq. (1) is one of the two

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integrable complexifications of the famous modified KdV equation, the other complexification (complex mKdV I) being simply $U_t = U_{xxx} + 6U\bar{U}U_x$.

In turn, the complex sine-Gordon II equation^{17,38} is a hyperbolic PDE for a complex function ψ of the form

$$\psi_{xy} = \frac{\bar{\psi}\psi_x\psi_y}{\psi\bar{\psi} + c} + (2\psi\bar{\psi} + c)(\psi\bar{\psi} + c)k\psi,$$

where c and k are arbitrary constants. The usual sine-Gordon equation $\phi_{xy} = c^2k \sin(\phi)$ is recovered upon setting $\psi = \bar{\psi} = \sqrt{-c} \sin(\phi/4)$.

The Sasa-Satsuma and the complex sine-Gordon II equation are of considerable interest for applications. The Sasa-Satsuma equation is widely used in nonlinear optics (see, e.g., Refs. 24 and 34 and references therein) because the integrable cases of the so-called higher order nonlinear Schrödinger equation²⁵ describing the propagation of short pulses in optical fibers are related through a gauge transformation either to the Sasa-Satsuma equation³⁶ or to the so-called Hirota equation.¹⁹

The complex sine-Gordon II equation, along with the Pohlmeyer-Lund-Regge model^{33,26,16,30} also known as the complex sine-Gordon I, defines integrable perturbations of conformal field theories^{12,2,8,3} (see, e.g., Refs. 35 and 37 for other applications). Moreover, the complex sine-Gordons I and II are the only equations for one complex field in the plane for which the (multi-)vortex solutions are found in closed form.^{31,5,6}

Following Ref. 21, we set $u = \psi$ and $v = \bar{\psi}$ and write the complex sine-Gordon II equation along with its complex conjugate as a system for u and v ,

$$u_{xy} = \frac{vu_xu_y}{uv + c} + (2uv + c)(uv + c)ku, \quad v_{xy} = \frac{uv_xv_y}{uv + c} + (2uv + c)(uv + c)kv. \quad (2)$$

Likewise, upon setting $p = U$ and $q = \bar{U}$ in the Sasa-Satsuma equation, proceeding in the same fashion as above and writing out $(pq)_x$ as $pq_x + p_xq$, we obtain

$$p_t = p_{xxx} + 9ppq_x + 3p^2q_x, \quad q_t = q_{xxx} + 9pqq_x + 3q^2p_x. \quad (3)$$

From now on we shall treat u , v , p , and q as independent variables that can be real or complex and consider systems (2) and (3) that are more general than the original complex sine-Gordon II and Sasa-Satsuma (complex mKdV II) equations which can be recovered under the reductions $v = \bar{u}$ and $q = \bar{p}$, respectively. In what follows we shall refer to Eq. (2) as to the *complex sine-Gordon II system* and to Eq. (3) as to the *Sasa-Satsuma system*.

Let us briefly address the relationship of the Sasa-Satsuma system [Eq. (3)] with other integrable systems. First of all, Eq. (3) can be obtained⁴⁶ as a reduction of the four-component Yajima-Oikawa system (Eq. 8 in Ref. 46).

On the other hand, consider the vector modified KdV equation,

$$V_t = V_{xxx} + \langle V, V \rangle V_x + \langle V, V_x \rangle V, \quad (4)$$

studied by Svinolupov and Sokolov⁴² (see also Refs. 43 and 45). Here, $V = (V^1, \dots, V^n)^T$ is an n -component vector, $\langle \cdot, \cdot \rangle$ stands for the usual Euclidean scalar product of two vectors, and the superscript T here and below indicates the transposed matrix. A simple linear change of variables $V^1 = \sqrt{3}(p+q)/\sqrt{2}$ and $V^2 = i\sqrt{3}(p-q)/\sqrt{2}$ takes Eq. (3) into Eq. (4) with $n=2$.

Setting $r = \sqrt{3}p$ and $s = \sqrt{3}q$ turns Eq. (3) into the system,

$$\begin{aligned} r_t &= r_{xxx} + r(3sr_x + rs_x), \\ s_t &= s_{xxx} + s(3rs_x + sr_x), \end{aligned} \quad (5)$$

studied by Foursov,¹⁵ who found a Hamiltonian structure for Eq. (5) of the form

$$\tilde{P} = \begin{pmatrix} -(1/3)rD_x^{-1} \circ r & D_x + (1/3)rD_x^{-1} \circ s \\ D_x + (1/3)sD_x^{-1} \circ r & -(1/3)sD_x^{-1} \circ s \end{pmatrix}, \quad (6)$$

where D_x is the operator of total x -derivative (see, e.g., Refs. 7, 11, and 32 for details and for the background on the recursion operators, Hamiltonian, and symplectic structures). The corresponding Hamiltonian density is¹⁵ $(2/3)r^2s^2 - r_x s_x$. This immediately yields a Hamiltonian structure [Eq. (8)] for Eq. (3). Note that in Refs. 14 and 15 there seems to be a misprint in \tilde{P} , and in Eq. (6) we corrected this misprint.

In p. 89 of Ref. 14, Foursov claims to have found some skew-symmetric operators that are likely to provide higher Hamiltonian structures for Eq. (5) but he failed to verify that these operators are indeed Hamiltonian. The explicit form of these operators was not presented in Refs. 15 and 14, so apparently it was never proved in the literature that Eq. (5) [and hence Eq. (3)] is bi-Hamiltonian systems. We establish the bi-Hamiltonian nature of Eq. (3) in Theorem 1 below.

Now turn to the complex sine-Gordon II system [Eq. (2)]. Recall that Eq. (2) can be obtained (see, e.g., Refs. 17, 21, and 28) as the Euler-Lagrange equation for the functional $S = \int L dx dy$, where

$$L = \frac{1}{2} \frac{u_x v_y + u_y v_x}{uv + c} + k(uv + c)uv.$$

A few conservation laws and generalized symmetries for Eq. (2) can be readily found, e.g., using computer algebra.^{21,28} In particular, Eq. (2) is compatible²¹ with

$$\begin{aligned} u_t &= u_{xxx} - \frac{3uv_x u_{xx}}{uv + c} - \frac{9u_x^2 v_x}{uv + c} + \frac{3u^2 v_x^2 u_x}{(uv + c)^2}, \\ v_t &= v_{xxx} - \frac{3vu_x v_{xx}}{uv + c} - \frac{9v_x^2 u_x}{uv + c} + \frac{3v^2 u_x^2 v_x}{(uv + c)^2}. \end{aligned} \quad (7)$$

The compatibility here means that the flow [Eq. (7)] commutes with the nonlocal flow [Eq. (14)] associated with Eq. (2) or, equivalently, the right-hand sides of Eq. (7) constitute the characteristic of a third-order generalized symmetry for Eq. (2) (see, e.g., Ref. 32 for general background on symmetries).

The rest of paper is organized as follows. In Sec. II we present symplectic structure and recursion operator for the Sasa-Satsuma system [Eq. (3)] and show that the latter is a bi-Hamiltonian system. In Sec. III we employ a nonlocal change of variables relating systems (7) and (3) in order to construct recursion operator and Hamiltonian and symplectic structures for Eqs. (2) and (7) from those of Eq. (3), and we show that Eq. (7) is a bi-Hamiltonian system. Finally, in Sec. IV we discuss the hierarchies of local and nonlocal symmetries for Eqs. (2), (3), and (7).

II. RECURSION OPERATOR AND SYMPLECTIC STRUCTURE FOR THE SASA-SATSUMA SYSTEM

A straightforward but tedious computation proves the following assertion.

Theorem 1: *The Sasa-Satsuma system [Eq. (3)] possesses a Hamiltonian structure,*

$$\mathcal{P} = \begin{pmatrix} -pD_x^{-1} \circ p & D_x + pD_x^{-1} \circ q \\ D_x + qD_x^{-1} \circ p & -qD_x^{-1} \circ q \end{pmatrix}, \quad (8)$$

a symplectic structure,

$$\mathcal{J} = \begin{pmatrix} 3pD_x^{-1} \circ p & D_x + 5pD_x^{-1} \circ q \\ D_x + 5qD_x^{-1} \circ p & 3qD_x^{-1} \circ q \end{pmatrix}, \quad (9)$$

and a hereditary recursion operator $\mathcal{R} = \mathcal{P} \circ \mathcal{J}$ that can be written as

$$\mathcal{R} = \begin{pmatrix} D_x^2 + 6pq + q_x D_x^{-1} \circ p + b D_x^{-1} \circ p + 3p_x D_x^{-1} \circ b & 2p^2 - 2z_1 p D_x^{-1} \circ q - 3p_x D_x^{-1} \circ a \\ 2q^2 - 2z_2 q D_x^{-1} \circ p + 3q_x D_x^{-1} \circ b & D_x^2 + 6pq + p_x D_x^{-1} \circ q + a D_x^{-1} \circ q - 3q_x D_x^{-1} \circ a \end{pmatrix}, \quad (10)$$

where $a = p_x + 2z_1 q$, $b = q_x + 2z_2 p$; $z_1 = D_x^{-1}(p^2)$ and $z_2 = D_x^{-1}(q^2)$ are potentials for the following conservation laws of Eq. (3):

$$D_t(p^2) = D_x(2pp_{xx} - p_x^2 + 6p^3q), \quad D_t(q^2) = D_x(2qq_{xx} - q_x^2 + 6pq^2).$$

Hence, Eq. (3) has an infinite hierarchy of compatible Hamiltonian structures $\mathcal{P}_k = \mathcal{R}^k \circ \mathcal{P}$, $k = 0, 1, 2, \dots$, $\mathcal{P}_0 \equiv \mathcal{P}$, an infinite hierarchy of symplectic structures $\mathcal{J}_k = \mathcal{J} \circ \mathcal{R}^k$, $k = 0, 1, 2, \dots$, and an infinite hierarchy of commuting symmetries of the form $\mathcal{K}_i = \mathcal{R}^i(\mathcal{K}_0)$, $i = 0, 1, 2, \dots$, where $\mathcal{K}_0 = (p_x, q_x)^T$.

The Sasa-Satsuma system [Eq. (3)] is bi-Hamiltonian with respect to \mathcal{P}_0 and \mathcal{P}_1 ,

$$\begin{pmatrix} p_t \\ q_t \end{pmatrix} = \mathcal{P}_0 \begin{pmatrix} \delta \mathcal{H}_1 / \delta p \\ \delta \mathcal{H}_1 / \delta q \end{pmatrix} = \mathcal{P}_1 \begin{pmatrix} \delta \mathcal{H}_0 / \delta p \\ \delta \mathcal{H}_0 / \delta q \end{pmatrix},$$

where $\mathcal{H}_i = \int H_i dx$, $i = 0, 1$, $H_0 = pq$, and $H_1 = 2p^2q^2 - p_x q_x$.

Here, $\delta / \delta p$ and $\delta / \delta q$ denote variational derivatives with respect to p and q .

Note that the Hamiltonian structure \mathcal{P} above is nothing but the second Hamiltonian structure of the Ablowitz-Kaup-Newell-Segur (AKNS) system (see, e.g., Ref. 27). This Hamiltonian structure (more precisely, its counterpart [Eq. (6)] for Eq. (5)) has already appeared in Refs. 14 and 15.

The symmetries \mathcal{K}_i commute because \mathcal{R} is hereditary. Applying \mathcal{R} to an obvious symmetry $\mathcal{K}_0 = (p_x, q_x)^T$ of Eq. (3) yields the symmetry $\mathcal{K}_1 = \mathcal{R}(\mathcal{K}_0) = (p_{xxx} + 9pqq_x + 3p^2q_x, q_{xxx} + 9pqq_x + 3q^2p_x)^T$, i.e., the right-hand side of Eq. (3). In turn, $\mathcal{R}(\mathcal{K}_1)$ is a local fifth-order symmetry for Eq. (3). We guess that \mathcal{K}_i are local for all natural i but so far we were unable to provide a rigorous proof of this.

Unlike the overwhelming majority of the hitherto known recursion operators (see, e.g., discussion in Ref. 39 and references therein), the nonlocal variables appear explicitly in the coefficients of \mathcal{R} . Perhaps this is the very reason why \mathcal{R} was not found earlier. The nonlocal variables in the coefficients of \mathcal{R} are Abelian pseudopotentials as in Ref. 23 and unlike, e.g., the nonlocalities in the recursion operator discovered by Karasu *et al.*²² and later rewritten in Ref. 40: the nonlocalities in the operator from Refs. 22 and 40 are non-Abelian pseudopotentials.

III. RECURSION OPERATOR FOR THE COMPLEX SINE-GORDON II SYSTEM

There is a well-known (see, e.g., Ref. 1 for discussion and references) transformation $z = \sqrt{2/3}g_x$ relating the symmetry flow $g_t = g_{xxx} + g_x^3/2$ of the sine-Gordon equation $g_{xy} = \sin(g)$ and the mKdV equation $z_t = z_{xxx} + z^2z_x$. Moreover, there exists¹ a nonlocal generalization of this transformation that sends the third-order symmetry flow,

$$u_t = u_{xxx} - 3 \frac{uv_x u_{xx}}{uv + c} + 3 \frac{(-uvu_x - cu_x + u^2v_x)u_x v_x}{(uv + c)^2},$$

$$v_t = v_{xxx} - 3 \frac{vu_x v_{xx}}{uv + c} - 3 \frac{(-v^2u_x + uvv_x + cv_x)u_x v_x}{(uv + c)^2},$$

of the complex sine-Gordon I equation (see, e.g., Refs. 16, 21, and 28)

$$u_{xy} = \frac{vu_x u_y}{uv + c} + (uv + c)ku, \quad v_{xy} = \frac{uv_x v_y}{uv + c} + (uv + c)kv, \quad (11)$$

into the two-component generalization of the mKdV equation,

$$p_t = p_{xxx} + 6pq p_x, \quad q_t = q_{xxx} + 6pqq_x,$$

that belongs to the hierarchy of the well-known AKNS system,

$$p_t = p_{xx} + p^2 q, \quad q_t = -q_{xx} - q^2 p. \quad (12)$$

Note that this nonlocal transformation also sends¹ the second-order symmetry flow,

$$u_t = u_{xx} - 2 \frac{uu_x v_x}{uv + c}, \quad v_t = -v_{xx} + 2 \frac{vu_x v_x}{uv + c},$$

of Eq. (11) into the AKNS system [Eq. (12)].

It turns out that upon a suitable redefinition of nonlocal variables, the nonlocal transformation in question in combination with a suitable rescaling of dependent variables p and q also sends the third-order symmetry flow [Eq. (7)] of the complex sine-Gordon II equation [Eq. (2)] into the Sasa-Satsuma system [Eq. (3)]. That is, we have the following result.

Theorem 2: *The substitution,*

$$p = \frac{i\sqrt{2}u_x \exp(-(1/2)w_1)}{\sqrt{uv + c}}, \quad q = \frac{i\sqrt{2}v_x \exp((1/2)w_1)}{\sqrt{uv + c}},$$

where $i = \sqrt{-1}$ and $w_1 = D_x^{-1}(\rho_1)$ is a potential for the conservation law $D_t(\rho_1) = D_x(\sigma_1)$ of Eq. (7),

$$\rho_1 = \frac{uv_x - vu_x}{uv + c}, \quad \sigma_1 = \frac{uv_{xxx} - vu_{xxx}}{uv + c} + \frac{(3uv + 2c)(v_x u_{xx} - u_x v_{xx})}{(uv + c)^2} + \frac{(12uv + 11c)(vu_x - uv_x)u_x v_x}{(uv + c)^3},$$

takes Eq. (7) into the Sasa-Satsuma system [Eq. (3)].

Here and below, D_x and D_t stand for the total x - and t -derivatives as defined, e.g., in Ref. 32.

Remark 1: *Let*

$$\tilde{p} = \frac{i\sqrt{2}u_x \exp(-(1/2)w_1)}{\sqrt{uv + c}}, \quad \tilde{q} = \frac{i\sqrt{2}v_x \exp((1/2)w_1)}{\sqrt{uv + c}},$$

where $w_1 = D_x^{-1}(\rho_1)$ is now a potential for the conservation law $D_y(\rho_1) = D_x(\theta_1)$ of Eq. (2), ρ_1 is given above, and

$$\theta_1 = \frac{vu_y - uv_y}{uv + c}.$$

Then for $k=0$, we have

$$D_y(\tilde{p}) = 0, \quad D_y(\tilde{q}) = 0,$$

where D_y stands for the total y -derivative. In other words, if $k=0$ then \tilde{p} and \tilde{q} provide nonlocal y -integrals for Eq. (2), and $D_x \ln(\tilde{p})$ and $D_x \ln(\tilde{q})$ are local y -integrals of Eq. (2). Therefore, the complex sine-Gordon II system [Eq. (2)] for $k=0$ is Liouvillean and C -integrable (see Ref. 9 and 10 for the construction of symmetries of such systems using the integrals thereof). Using the above local integrals enables us to find the general solution for Eq. (2) with $k=0$ along the lines of Ref. 47.

Passing from p and q to u and v yields from \mathcal{R} a recursion operator \mathfrak{R}_0 for Eq. (7). The operator \mathfrak{R}_0 is hereditary because so is \mathcal{R} (see Ref. 13). It is easily seen that \mathfrak{R}_0 is a recursion operator for Eq. (2) as well. Upon removing an inessential overall constant factor in \mathfrak{R}_0 , we obtain the following result.

Theorem 3: *The complex sine-Gordon II system [Eq. (2)] has a hereditary recursion operator,*

$$\mathfrak{R} = \begin{pmatrix} D_x^2 - \frac{2uv_x}{uv+c} + \frac{uv_{xx} - 12u_x v_x}{uv+c} + \frac{cu_x v_x}{(uv+c)^2} & -\frac{2uu_{xx} + 6u_x^2}{uv+c} + \frac{4u^2 u_x v_x}{(uv+c)^2} \\ -\frac{2vv_{xx} + 6v_x^2}{uv+c} + \frac{4v^2 u_x v_x}{(uv+c)^2} & D_x^2 - \frac{2vu_x}{uv+c} + \frac{vu_{xx} - 12u_x v_x}{uv+c} + \frac{cu_x v_x}{(uv+c)^2} \end{pmatrix} \\ + \begin{pmatrix} \sum_{\alpha=1}^5 Q_\alpha^1 D_x^{-1} \circ \gamma_{\alpha,1} & \sum_{\alpha=1}^5 Q_\alpha^1 D_x^{-1} \circ \gamma_{\alpha,2} \\ \sum_{\alpha=1}^5 Q_\alpha^2 D_x^{-1} \circ \gamma_{\alpha,1} & \sum_{\alpha=1}^5 Q_\alpha^2 D_x^{-1} \circ \gamma_{\alpha,2} \end{pmatrix},$$

where w_1 is as in Remark 1; $y_1 = D_x^{-1}[(u_x^2 \exp(-w_1))/(uv+c)]$ and $y_2 = D_x^{-1}[(v_x^2 \exp(w_1))/(uv+c)]$ are potentials for the nonlocal conservation laws $D_y(\zeta_i) = D_x(\xi_i)$, $i=1, 2$, of Eq. (2),

$$\zeta_1 = \frac{u_x^2 \exp(-w_1)}{uv+c}, \quad \xi_1 = ku^2(uv+c)\exp(-w_1),$$

$$\zeta_2 = \frac{v_x^2 \exp(w_1)}{uv+c}, \quad \xi_2 = kv^2(uv+c)\exp(w_1);$$

$$Q_1 = \begin{pmatrix} y_1 u \\ u_x \exp(-w_1) - v y_1 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} v_x \exp(w_1) - u y_2 \\ y_2 v \end{pmatrix},$$

$$Q_3 = \begin{pmatrix} u \\ -v \end{pmatrix}, \quad Q_4 = c \begin{pmatrix} -u_{xx} + \frac{2uu_x v_x}{uv+c} + 4 \exp(w_1) y_1 v_x - 4u y_2 y_1 \\ v_{xx} - \frac{2vu_x v_x}{uv+c} - 4 \exp(w_1) y_2 u_x + 4v y_2 y_1 \end{pmatrix}, \quad Q_5 = \begin{pmatrix} u_x \\ v_x \end{pmatrix}$$

are symmetries for Eq. (2), and

$$\gamma_1 = \left(\frac{\exp(w_1) v v_x^2 + c y_2 v_x}{(uv+c)^2}, \frac{\exp(w_1) v_{xx}}{uv+c} - \frac{2 \exp(w_1) v u_x v_x + c y_2 u_x}{(uv+c)^2} \right),$$

$$\gamma_2 = \left(\frac{\exp(-w_1) u_{xx}}{uv+c} - \frac{2 \exp(-w_1) u u_x v_x + c y_1 v_x}{(uv+c)^2}, \frac{\exp(-w_1) u u_x^2 + c y_1 u_x}{(uv+c)^2} \right)$$

$$\gamma_3 = \left(-\frac{v_{xxx}}{uv+c} + \frac{2(vv_x + 2 \exp(-w_1) u v y_2 + 2 \exp(-w_1) c y_2) u_{xx} + (2v u_x + u v_x) v_{xx}}{(uv+c)^2} \right. \\ \left. + \frac{2v_x u_x (-v^2 u_x + 2c v_x)}{(uv+c)^3} - \frac{2v_x (-u_x v_x + 4 \exp(-w_1) u y_2 u_x + 2 \exp(w_1) v y_1 v_x + 2c y_2 y_1)}{(uv+c)^2} \right. \\ \left. - \frac{u_{xxx}}{uv+c} - \frac{2(uu_x + 2 \exp(w_1) u v y_1 + 2 \exp(w_1) c y_1) v_{xx} - (2v u_x + v u_x) u_{xx}}{(uv+c)^2} \right. \\ \left. - \frac{2v_x u_x (-u^2 v_x + 2c u_x)}{(uv+c)^3} + \frac{2u_x (-u_x v_x + 4 \exp(w_1) v y_1 v_x + 2 \exp(-w_1) u y_2 u_x + 2c y_2 y_1)}{(uv+c)^2} \right),$$

$$\gamma_4 = \left(\frac{v_x}{(uv+c)^2}, -\frac{u_x}{(uv+c)^2} \right), \quad \gamma_5 = \left(\frac{v_{xx}}{(uv+c)} - \frac{u v_x}{(uv+c)^2}, \frac{u_{xx}}{(uv+c)} - \frac{v u_x}{(uv+c)^2} \right)$$

are cosymmetries for Eq. (2).

Here, Q_α^i and $\gamma_{\alpha,j}$ denote the i th component of Q_α and j th component of γ_α , respectively, i.e.,

$$Q_\alpha \equiv \begin{pmatrix} Q_\alpha^1 \\ Q_\alpha^2 \end{pmatrix}, \quad \gamma_\alpha \equiv (\gamma_{\alpha,1}, \gamma_{\alpha,2}).$$

Note that if we use the tensorial notation (see, e.g., Ref. 28), we can rewrite the operator

$$\begin{pmatrix} \sum_{\alpha=1}^5 Q_\alpha^1 D_x^{-1} \circ \gamma_{\alpha,1} & \sum_{\alpha=1}^5 Q_\alpha^1 D_x^{-1} \circ \gamma_{\alpha,2} \\ \sum_{\alpha=1}^5 Q_\alpha^2 D_x^{-1} \circ \gamma_{\alpha,1} & \sum_{\alpha=1}^5 Q_\alpha^2 D_x^{-1} \circ \gamma_{\alpha,2} \end{pmatrix}$$

in a more concise form, namely, $\sum_{\alpha=1}^5 Q_\alpha \otimes D_x^{-1} \circ \gamma_\alpha$.

Remark 2: Equation (7) has a recursion operator of precisely the same form as the \mathfrak{R} given above, but in this case the nonlocal variables should be defined in a slightly different way: w_1 should be as in Theorem 2, and $y_1 = D_x^{-1}((u_x^2 \exp(-w_1))/(uv+c))$ and $y_2 = D_x^{-1}((v_x^2 \exp(w_1))/(uv+c))$ should now be potentials for the nonlocal conservation laws $D_x(\zeta_i) = D_x(\chi_i)$, $i=1, 2$, of Eq. (7), where ζ_i are as in Theorem 1 and

$$\chi_1 = \exp(-w_1) \left(\frac{2u_x u_{xxx}}{uv+c} - \frac{u_{xx}^2}{uv+c} - \frac{2uu_x v_x u_{xx}}{(uv+c)^2} - \frac{2uu_x^2 v_{xx}}{(uv+c)^2} - \frac{2(6uv+7c)u_x^3 v_x}{(uv+c)^3} + \frac{3u^2 u_x^2 v_x^2}{(uv+c)^3} \right),$$

$$\chi_2 = \exp(w_1) \left(\frac{2v_x v_{xxx}}{uv+c} - \frac{v_{xx}^2}{uv+c} - \frac{2vv_x^2 u_{xx}}{(uv+c)^2} - \frac{2vv_x v_x v_{xx}}{(uv+c)^2} + \frac{3v^2 u_x^2 v_x^2}{(uv+c)^3} - \frac{2(6uv+7c)u_x v_x^3}{(uv+c)^3} \right).$$

Recall that Eq. (2) possesses^{28,29} a local symplectic structure,

$$\mathfrak{J} = \begin{pmatrix} 0 & \frac{1}{uv+c} D_x - \frac{uv_x}{(uv+c)^2} \\ \frac{1}{uv+c} D_x - \frac{vu_x}{(uv+c)^2} & 0 \end{pmatrix},$$

i.e., Eq. (2) can be written in the form $\mathfrak{J}u_y = k(2uv+c)(v,u)^T = \delta\mathcal{H}/\delta u$, where $u=(u,v)^T$ and $\mathcal{H} = \int k(uv+c)uv dx$. The symplectic structure \mathfrak{J} is readily seen to be shared by Eq. (7).

Inverting \mathfrak{J} yields a nonlocal Hamiltonian structure for Eq. (7) of the form

$$\mathfrak{P} = \begin{pmatrix} 0 & \exp(-w_1/2)\sqrt{uv+c} \circ D_x^{-1} \circ \exp(w_1/2)\sqrt{uv+c} \\ \exp(w_1/2)\sqrt{uv+c} \circ D_x^{-1} \circ \exp(-w_1/2)\sqrt{uv+c} & 0 \end{pmatrix},$$

where w_1 should be interpreted as in Theorem 2. It can be shown that if we go from the (u,v) to the (p,q) variables using the transformation from Theorem 2, then we obtain from \mathfrak{P} the Hamiltonian structure [Eq. (8)] for the Sasa-Satsuma system up to an inessential overall constant factor.

The higher Hamiltonian (resp. symplectic) structures for Eq. (7) are given by the formulas $\mathfrak{P}_n = \mathfrak{R}^n \circ \mathfrak{P}$ (resp. $\mathfrak{J}_n = \mathfrak{J} \circ \mathfrak{R}^n$), where $n=1, 2, 3, \dots$. The operator \mathfrak{R} is hereditary, and hence all of these structures are compatible. However, the structures \mathfrak{P}_1 and \mathfrak{J}_1 are already very cumbersome, so we do not display them here. Nevertheless, it is readily checked that the following assertion holds.

Theorem 4: Equation (7) is a bi-Hamiltonian system,

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \mathfrak{P}_0 \begin{pmatrix} \delta\mathfrak{H}_1/\delta u \\ \delta\mathfrak{H}_1/\delta v \end{pmatrix} = \mathfrak{P}_1 \begin{pmatrix} \delta\mathfrak{H}_0/\delta u \\ \delta\mathfrak{H}_0/\delta v \end{pmatrix},$$

where $\mathfrak{H}_i = \int \tilde{H}_i dx$, $i=0, 1$,

$$\tilde{H}_0 = -\frac{u_x v_x}{uv+c}, \quad \tilde{H}_1 = \frac{u_{xx} v_{xx}}{uv+c} - \frac{v u_x v_x u_{xx} + u u_x v_x v_{xx} - 4u_x^2 v_x^2}{(uv+c)^2} + \frac{uv u_x^2 v_x^2}{(uv+c)^3}.$$

It is straightforward to verify that the action of \mathfrak{R} on the obvious symmetry \mathbf{u}_x of Eq. (7) yields the right-hand side of Eq. (7), and $\mathfrak{R}^2(\mathbf{u}_x)$ is a fifth-order *local* generalized symmetry for Eq. (2). We guess that the repeated application of \mathfrak{R} to \mathbf{u}_x yields a hierarchy of *local* generalized symmetries for Eq. (2) but we were not able to prove this in full generality so far, as the presence of the nonlocalities y_i in the coefficients of \mathfrak{R} appears to render useless the hitherto known ways of proving locality for hierarchies of symmetries generated by the recursion operator (cf., e.g., Ref. 41 and references therein). Nevertheless, as \mathfrak{R} is a recursion operator for Eq. (7), Proposition 2 from Ref. 39 tells us that the only nonlocalities that could possibly appear in the hierarchy of the symmetries $\mathfrak{R}^k(\mathbf{u}_x)$, $k=1, 2, \dots$, are potentials of (possibly nonlocal) conservation laws for Eq. (2).

IV. NONLOCAL SYMMETRIES

First of all, we can consider Eq. (2) as a nonlocal symmetry flow of Eq. (7). We have already noticed above that Eq. (2) can be written as

$$\mathfrak{J} \mathbf{u}_y = \delta \mathfrak{H} / \delta \mathbf{u}, \quad (13)$$

where $\mathfrak{H} = \int k(uv+c)uv dx$. Acting by $\mathfrak{P} = \mathfrak{J}^{-1}$ on both sides of Eq. (13), we can formally rewrite the complex sine-Gordon II system [Eq. (2)] in the evolutionary form $\mathbf{u}_y = \mathfrak{P}(\delta \mathfrak{H} / \delta \mathbf{u})$, that is,

$$u_y = k \exp(-w_1/2) \sqrt{uv+c} \omega_2, \quad v_y = k \exp(w_1/2) \sqrt{uv+c} \omega_1, \quad (14)$$

where $\omega_1 = D_x^{-1}(v(2uv+c)\exp(-w_1/2)\sqrt{uv+c})$ and $\omega_2 = D_x^{-1}(u(2uv+c)\exp(w_1/2)\sqrt{uv+c})$ are potentials for the following nonlocal conservation laws of Eq. (7):

$$D_t(v(2uv+c)\exp(-w_1/2)\sqrt{uv+c}) = D_x \left(\exp(-w_1/2)\sqrt{uv+c} \left(\frac{(4uv+3c)v^2 u_{xx}}{uv+c} + (4uv+c)v_{xx} - \frac{2v^3 u_x^2}{uv+c} - \frac{v(20u^2 v^2 + 26cuv + 7c^2)u_x v_x}{(uv+c)^2} - 2uv_x^2 \right) \right),$$

$$D_t(u(2uv+c)\exp(w_1/2)\sqrt{uv+c}) = D_x \left(\exp(w_1/2)\sqrt{uv+c} \left((4uv+c)u_{xx} + \frac{u^2(4uv+3c)v_{xx}}{uv+c} - 2vu_x^2 - \frac{u(20u^2 v^2 + 26cuv + 7c^2)u_x v_x}{(uv+c)^2} - \frac{2u^3 v_x^2}{uv+c} \right) \right).$$

Equation (14) is compatible with Eq. (7), i.e., the right-hand side of Eq. (14) divided by k ,

$$\mathbf{Q}_{-1} = \begin{pmatrix} \exp(-w_1/2)\sqrt{uv+c}\omega_2 \\ \exp(w_1/2)\sqrt{uv+c}\omega_1 \end{pmatrix},$$

is a nonlocal symmetry for Eq. (7). Moreover, we have $\mathfrak{R}(\mathbf{Q}_{-1}) = c^2 \mathbf{u}_x$, i.e., the action of \mathfrak{R} on \mathbf{Q}_{-1} gives the “zeroth” (obvious) symmetry \mathbf{u}_x up to a constant factor. Thus, Eq. (14) (and hence the complex sine-Gordon II system [Eq. (2)]) can be considered as a first negative flow in the hierarchy of Eq. (7). Note that if we consider \mathfrak{R} as a recursion operator for Eq. (2), we have $\mathfrak{R}(\mathbf{u}_y) = kc^2 \mathbf{u}_x$, in perfect agreement with the above result.

Equations (3) and (2) possess nonlocal symmetries of the form

$$\mathcal{G}_1 = \begin{pmatrix} q_x + z_2 p \\ -z_2 q \end{pmatrix}, \quad \mathcal{G}_2 = \begin{pmatrix} -z_1 p \\ p_x + z_1 q \end{pmatrix},$$

and

$$\mathcal{Q}_1 = \begin{pmatrix} y_1 u \\ u_x \exp(-w_1) - v y_1 \end{pmatrix}, \quad \mathcal{Q}_2 = \begin{pmatrix} v_x \exp(w_1) - u y_2 \\ y_2 v \end{pmatrix},$$

respectively, where z_1 and z_2 were defined in Theorem 1. Therefore, we have four hierarchies of nonlocal symmetries: $\mathcal{R}^i(\mathcal{G}_j)$ for Eq. (3) and $\mathfrak{R}^i(\mathcal{Q}_j)$ for Eq. (2), where in both cases $j=1, 2$ and $i=0, 1, 2, \dots$

Two more hierarchies of nonlocal symmetries have the form $\mathcal{R}^i(\mathcal{G}_0)$ for Eq. (3) and $\mathfrak{R}^i(\mathcal{Q}_0)$ for Eq. (2), where $i=1, 2, \dots$,

$$\mathcal{G}_0 = \begin{pmatrix} p \\ -q \end{pmatrix}, \quad \mathcal{Q}_0 = \begin{pmatrix} u \\ -v \end{pmatrix}.$$

Two somewhat more “usual” nonlocal hierarchies of master symmetries [the latter are, roughly speaking, time-dependent symmetries such that repeatedly commuting them with a suitable time-independent symmetry yields an infinite hierarchy of *time-independent* commuting symmetries for the system in question (see, e.g., Ref. 7 and references therein for further details)], $\mathcal{R}^i(\mathcal{S}_0)$ for Eq. (3) and $\mathfrak{R}^i(\mathcal{S}_0)$ for Eq. (7), where $i=1, 2, \dots$, originate from the scaling symmetries for Eqs. (3) and (7),

$$\mathcal{S}_0 = \begin{pmatrix} 3t(p_{xxx} + 9pqq_x + 3p^2q_x) + xp_x + p \\ 3t(q_{xxx} + 9pqq_x + 3q^2p_x) + xq_x + q \end{pmatrix}$$

and

$$\mathcal{S}_0 = \begin{pmatrix} 3t \left(u_{xxx} - \frac{3uw_x u_{xx}}{uv+c} - \frac{9u_x^2 v_x}{uv+c} + \frac{3u^2 v_x^2 u_x}{(uv+c)^2} \right) + xu_x \\ 3t \left(v_{xxx} - \frac{3vu_x v_{xx}}{uv+c} - \frac{9v_x^2 u_x}{uv+c} + \frac{3v^2 u_x^2 v_x}{(uv+c)^2} \right) + xv_x \end{pmatrix},$$

respectively.

Finally, Eq. (7) has nonlocal symmetries,

$$\mathcal{G}_1 = \begin{pmatrix} \sqrt{uv+c} \exp(-w_1/2) \\ 0 \end{pmatrix}, \quad \mathcal{G}_2 = \begin{pmatrix} 0 \\ \sqrt{uv+c} \exp(w_1/2) \end{pmatrix},$$

obtained by differentiating \mathcal{Q}_{-1} with respect to ω_1 and ω_2 , but these symmetries are annihilated by \mathfrak{R} and hence do not lead to new hierarchies of nonlocal symmetries.

It would be interesting to find out whether the systems in question possess nonlocal symmetries that do not belong to the above hierarchies, which is the form of solutions invariant under the nonlocal symmetries and whether these solutions could have any applications in nonlinear optics.

As a final remark, we note that because of the obvious symmetry of the complex sine-Gordon II system [Eq. (2)] under the interchange of x and y , all of the above results concerning the recursion operator, Hamiltonian and symplectic structures, and hierarchies of symmetries for Eq. (2) remain valid if we replace all x -derivatives by y -derivatives and vice versa and swap the operators D_x and D_y . Interestingly enough, the recursion operator $\tilde{\mathfrak{R}}$ obtained from \mathfrak{R} upon such an interchange proves to be inverse to \mathfrak{R} on symmetries of Eq. (2) up to a constant factor. More precisely, for any symmetry K of Eq. (2) we have

$$\tilde{\mathfrak{R}}(\mathbf{K}) = k^2 c^4 \mathfrak{R}^{-1}(\mathbf{K}).$$

Taking into account our earlier results, we see that the “basic” hierarchy of Eq. (2) can be represented by a diagram of the form

$$\cdots \rightarrow \mathfrak{R}^{-2}(\mathbf{u}_y) \rightarrow \mathfrak{R}^{-1}(\mathbf{u}_y) \rightarrow \mathbf{u}_y \rightarrow \mathbf{u}_x = \mathfrak{R}(\mathbf{u}_y)/kc^2 \rightarrow \mathfrak{R}(\mathbf{u}_x) \rightarrow \mathfrak{R}^2(\mathbf{u}_x) \rightarrow \cdots,$$

and we guess that *all* symmetries presented at this diagram are *local*, i.e., they do not involve nonlocal variables. This is a fairly common situation for hyperbolic PDEs (see, e.g., the discussion in Ref. 4).

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