The Dark Quantum States of Gravity

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Abstract
Quantum theory applied to gravitational potentials predicts the existence of certain stable, macroscopic stationary state solutions that intrinsically possess all the physical properties required for dark matter, eliminating the necessity to introduce new particles or new physics. Traditional baryonic material occupying such states will be both stable and weakly interacting. These WIMP-like macroscopic quantum structures function as dark matter candidates for LCDM cosmology on the largest scales where it has been most successful, but retain potential to yield observationally compliant predictions on galactic cluster and sub-cluster scales. Relatively pure, high angular momentum eigenstate solutions form the structural basis of this quantum approach. They are seen to have no classical analogue, and properties radically different to those of traditionally localized matter or orbiting particles. Salient features of some of the solutions include long radiative lifetimes and energies and ‘sizes’ consistent with that expected for galactic halos. This facilitates the existence of sparsely populated, highly stable structures with negligible electromagnetic emission and inherent inability to gravitationally collapse.

Introduction
A wealth of astronomical observations suggests a universe dominated by unknown particles. The currently favoured explanation for this Dark Matter is Lambda Cold Dark Matter (LCDM) cosmology. LCDM has had excellent success on the larger scales but the difficulties that is has encountered on smaller scales suggest that some modification is needed at the cluster level and below. A potential solution to the problems on small scales can be obtained using a quantum approach to LCDM on the galactic scale. In particular in this paper we use the quantised solutions to a gravitational central potential in weak gravity developed by (Ernest 2003, 2005) and see how their properties differ from the atomic case. In particular we look at the decay times and stimulated transition rates and examine how they function as dark matter candidates.

A Weak Gravity Quantum Equation
It is a simple matter to recast the Schrödinger equation into a form suitable for weak gravity.

\[ \frac{-\hbar^2}{2\mu} \nabla^2 \psi - \frac{Gm_p M}{r} \psi = \frac{i\hbar}{\partial t} \partial \psi \]  \hspace{1cm} (1)

where \( M \) is the central mass, \( \mu \) is the reduced mass and the other symbols have their normal meanings. The solutions are analogous to those of the hydrogen atom and may be immediately written down, the eigenvalues \( E_n \) being

\[ E_n = -\frac{\mu G^2 m_p^2 M^2}{2\hbar^2 n^2} \]  \hspace{1cm} (2)

With the introduction of the parameter \( b_0 = \frac{\hbar^2}{G\mu m_p M} \), the corresponding eigenfunctions \( u_n(r,t) \) are

\[ u_{n,l,m}(r,t) = R_{n,l}(r)Y_{l,m}(\theta,\phi) \]  \hspace{1cm} (3)

where \( Y_{l,m}(\theta,\phi) \) are the normalized spherical harmonics and

\[ R_{n,l}(r) = N_{n,l} \left( \frac{2r}{n b_0} \right)^\frac{l}{2} \exp \left( -\frac{r}{n b_0} \right) L_{n-l-1}^{2l+1} \left( \frac{2r}{n b_0} \right) \]  \hspace{1cm} (4)

In equation (4) \( N_{n,l} = \left\{ \frac{2}{n b_0} \right\}^{\frac{3}{2}} \frac{(n-l-1)!}{2n(n+l)!} \) is a normalizing constant and

\[ L_{n-l-1}^{2l+1} \left( \frac{2r}{n b_0} \right) = (n+l)! \sum_{k=0}^{n-l-1} \frac{(-1)^{k+1}}{(n-l-1-k)! (2l+1+k)!} \]  \hspace{1cm} (5)
are the generalized Laguerre polynomials in their standard form.

It would be expected that the properties of the solutions would not be very different from those in the atomic case. However, large \( n, l \) valued eigenfunctions in the weak gravity regions of deep (large \( M \)) gravitational wells have structural properties that can exert major influences on interaction rates irrespective of the interaction Hamiltonian. We represent the states in fig. 1, where a new quantum parameter \( p (= n - l) \) has been introduced. \( n \) values are shown vertically on the diagram, \( l \) values horizontally and the diagonal lines represent sets of states having a constant \( p \) value. The significance of \( p \) is that it represents states of common “shape”, that is \( p \) is the number of peaks in the radial eigenfunction. Since for any fixed values for \( l \) and \( n \) the sum over all possible \( z \)-projection substates \( m \) yield a spherically symmetric probability density, each dot on fig. 1 represents \( 2l + 1 \) states which taken together produce a combined spherically symmetric probability density.

In the case of a gravitational atom (consisting of two neutral particles of \( \sim 1 \) amu) the \( n=1 \) binding energy is \( \sim 10^{-69} \) eV. This binding energy increases with larger mass but the state size becomes unrealistically small. Realistic binding energies are possible with larger central potential masses for high quantum numbers. These states are often in the macroscopic domain, but there is no evidence from quantum mechanics to deny their existence. Even decoherence, which limits the observation of large-mass interference effects, affects only the phase of an eigenstate (Reynaud 2001). Additionally it should be noted in the macroscopic domain one must distinguish between macroscopic orbiting bodies whose net eigenspectra clearly yield time dependent position probabilities and eigenstates which have no such dependence. (Localized particles necessarily involve eigenspectra which consist of the sum of a large number of delocalised states, the tighter the localisation constraint the greater the spread in the eigenspectra). The study of the theoretical properties of pure gravitational eigenstates requires examining states with large quantum numbers \( l, m, n, \) that fit into the weak gravity restriction imposed on the Schrödinger equation above.

**Eigenstate Decay Times**

We now consider the dipole decay rates of the various states on fig. 1 that are important in determining whether a state will be stable or radiatively decay. In particular we are concerned with electromagnetic decay and charged, rather than neutral particles occupying the eigenstates because it has been shown elsewhere that the decay due to gravitational radiation is negligible by comparison (Ernest 2005). The dipole radiative decay rate of any eigenstate is related to the transition probability \( A_{if} (\equiv \text{Einstein A coefficient}) \). For a given transition \( n_i \rightarrow n_f \) from initial state \( |i\rangle \) to final state \( |f\rangle \) is given by (Corney 1986)

\[
A_{if} = \frac{\omega_f^\prime \Pi_f}{3 \epsilon_0 c \hbar^3} = \frac{\omega_f \Pi_f^\prime}{3 \epsilon_0 c \hbar^3} \tag{6}
\]

where \( \omega_f^\prime = \omega_f + \frac{\mu G^2 m_p M^2}{2 \hbar^2} (\frac{1}{n_f^2} - 1/n_i^2) \) is the angular frequency corresponding to the transition \( i \rightarrow f \), \( \mu \) is the reduced mass, \( \Pi_f \) is the corresponding dipole matrix element for the transition \( i \rightarrow f \), and the other symbols have their normal meanings. The value of \( \Pi_f \) is given by

\[
\Pi_f = \int_0^{2\pi} \int_0^\pi \int_0^\pi R_{nf,c}^* R_{nf,cm}^* Y_{l,n}^* r^2 \sin^2(\theta) \cos(\phi) d\phi d\theta dr \tag{7}
\]

where \( \Pi_{if,c} \), \( \Pi_{if,y} \) and \( \Pi_{if,z} \) are the \( x, y, \) and \( z \) components of the vector inside the modulus in (7). Dipole radiative decay occurs via transitions involving \( \Delta m = 0 \) (implying \( \Pi_{if} = \Pi_{if,c} = 0 \)) or \( \Delta m = \pm 1 \) (implying \( \Pi_{if,c} = 0 \)) and \( \Delta l = \pm 1 \) (implying transitions must take place between adjacent \( l \) columns in fig. 1). \( \Pi_{if,c} \), \( \Pi_{if,y} \) and \( \Pi_{if,z} \) may be explicitly written as

\[
\Pi_{if,c} = e^{il} R_{nf,c}^* r^3 R_{ni,cm} dr \int_0^{2\pi} \int_0^\pi Y_{l,n}^* Y_{l,m} \sin(\theta) \cos(\phi) d\phi d\theta = e^{il} I_R I_{\theta\phi} \tag{8}
\]
with corresponding expressions for $y$ and $z$, where we have further split the integrals into their radial
\( I_R = \int_0^\infty R_{nf, l}(r) R_{nl, l}(r) r^3 dr \) and angular \( I_{\theta \phi} = \int_0^{2\pi} Y_{l, m}^* \sin(\theta) \cos(\theta) d\theta d\phi \) components.

For large values of $n$ the form of the eigenstates in general becomes very complex, often involving sums over an intractable number of terms. This presents a problem for the ultimate calculation of any specific transition rate if it involves many eigenfunction terms. Furthermore the calculation of any specific decay rate may involve the summation over any extremely large number of individual transition decay channels each of which may be very difficult to calculate. This may be simplified to some extent by including only those decay channels which do not involve non-overlapping states or channels where the structure of the two eigenfunctions is such that, although there is overlap, the overlap integral for that particular transition channel will be negligible (Ernest 2008).

When $l$ is small and $n$ is large then the radial component of the radial function has a large radial extent, while when $l \sim n$, the radial eigen function component is more localized in the $r$ direction, with the relative degree of localization increasing with increasing $n$. In the polar direction the spread of the wave function is more localized when $m \sim l$, and spread over a larger range of $\theta$ when $m$ is small compared to $l$. Spread out functions potentially have more overlap, and there are potentially many more non-negligible transitions. Of interest here are those functions which have limited radial extent (although the angular part can be spread). Such functions are represented by diagonal lines of low $p$ on fig. 1. These states are interesting because depending on the value on $n$ they can have long radiative lifetimes.

It has been shown in (Ernest 2008) that, for radiative dipole decay, the absolute angular components of the dipole matrix element have a limited size and depending on the relative value of $l$ and $m$ converge to small numbers or zero as $l$ and $m$ become large. Critical in determining the decay rate is value of radial component of $\Pi_{nf}$. For states on fig. 1 which have large $p$ values many decay channels are available and since large $p$ corresponds to lower $l$ values there will be considerable overlap in the functions involved in the overlap integral, thereby resulting in those channels having greater decay rates. However, for the low $p$ states there are fewer channels available and additionally, the overlap is much more limited and can essentially be zero.

As an example we concentrate in this paper on the lifetimes of the $p=1$ diagonal. For any state $n, l, m$ on this diagonal (where $l = n - 1$) there are only three dipole channels available for transfer $(n-1, l-1, m-1), (n-1, l-1, m), (n-1, l-1, m+1)$. The lifetime of these states may be considered for various values of $n$ ensuring that for each $n$ the central mass is sufficiently large to maintain a reasonable binding energy. In figures 2 and 3 we show a plot of the decay time for the electromagnetic dipole decay as function of $n$ value for the $p=1$ state with reasonable binding energies of 1 eV, 0.1 eV and 0.01 eV. We take reasonable binding energies to mean ones that are large compared to the magnitude of typical random background field fluctuations (e.g. $E \sim 10^{-3}$ eV for the cosmic background radiation).

The rise in decay time with size and mass might be expected given that high lying Rydberg state in atoms have relatively long lifetimes. However, what is interesting here is that these lifetimes continue to increase with larger central potential masses to the point where large states in deep potential wells have lifetimes exceeding the age of the universe. Significantly this long lifetime results in an inability for such states to gravitationally collapse. Although the decay times of these states are long one might expect that they could be excited to higher levels by stimulated absorption processes. The probability $P_{nf}$ of these types of processes is given by

![Figure 2: Decay time vs. radial position of eigenstate for fixed eigenstate energies $E$; from lowest to highest curve, $E = 10^2, 10^3, 10^4$ eV. The log-log plot appears linear because $\Pi_{nf} \propto n^2$ approximately.](image)

![Figure 3: Decay time vs. central mass for fixed eigenstate energies $E$; from lowest to highest curve, $E = 10^2, 10^3, 10^4$ eV. The log-log plot appears linear because $\Pi_{nf} \propto n^2$ approximately.](image)
\[
\Pi_{2f} = \frac{\pi \epsilon^2}{3\epsilon_0 \hbar^2} \langle f | \mathbf{r} | i \rangle^2 \rho (\omega_i) \tag{9}
\]

where \( \epsilon \langle f | \mathbf{r} | i \rangle \) is the dipole matrix element \( \Pi_{if} \) and \( \rho (\omega) \) is the spectral energy density per unit angular frequency (Corney 1986). This leads to the Einstein \( B \) coefficient given by

\[
B_{2f} = \frac{\pi \epsilon^2}{3\epsilon_0 \hbar^2} g_i \sum_{m_i,m_f} \left| \langle m_k | \mathbf{r} | m_i \rangle \right|^2 = \frac{c^3 \pi^2}{\hbar} \frac{g_f}{g_i} A_{f,i} \tag{10}
\]

where \( g_i \) and \( g_f \) are respectively the degeneracy of the two levels. For a constant \( p \) transition where \( \Delta n = \pm 1 \) and \( p = 1 \) the absorption probability per unit time for the relevant part of cosmic microwave background (CMB) is of the order of \( \sim 5 \times 10^{-2} \text{s}^{-1} \). At this level of CMB radiation the change in principle quantum number over a billion years would only be \( \sim 10^{15} \) which is negligible for 1 eV states in a central potential mass of \( 10^{42} \) kg.

For stimulated transitions where \( \Delta n > 1 \) it might be expected that the stimulated rate would be much greater than for the \( n = 1 \) case. This is because the higher \( n \) transitions correspond to frequencies where the CMB photon density is higher, and also because the factor of \( A_{if} \) in eqn (10) is expected to be much larger, since it goes as \( \omega^3 \). However, it was shown in (Ernest 2008) that the radial component of \( \Pi_{if} \) for \( p = 2 \) is less than that for \( p = 1 \) by a factor of \( \sqrt{2n_i} \). A similar trend continues: while \( p << n \) large, the ratio \( I_R(p+1)/I_R(p) \), is given approximately by \( I_R \sim \left( \frac{p}{2n_i} \right)^{(p+1)/p} / \left( \frac{p}{2n_j} \right)^{(p+1)/p} \sim \left( \frac{p}{2n_i} \right)^{(p+1)/p} / \left( \frac{p}{2n_j} \right)^{(p+1)/p} \) and demonstrates that \( I_R \) decreases dramatically (because of the large values of \( n_i \) and \( n_j \)) for each successive unit increase in \( p \). This results in the successive values for the overlap integral \( \Pi_{if} \) also being reduced by successive factors of \( \sim \sqrt{p/2n_i} \). This means that the rate of stimulated transitions to higher \( n \) levels decreases rapidly, despite increases in \( \omega \) and CMB photon density. For example at the peak present day CMB background the photon energy density is \( 2.6 \times 10^{-25} \text{ J m}^{-3} \text{srad}^{-1} \), which corresponds to a transition jump of \( \Delta p = 6 \times 10^{29} \) for a central mass potential of \( 10^{42} \) kg and a proton occupying the eigen state. Using equation (10) the transition rate to the upper state corresponding to the peak frequency of the present day CMB is of the order \( 10^{29} \) s\(^{-1}\). These examples illustrate that particles occupying high \( n \) low \( p \) gravitational eigenstates will be stable not only with respect to radiative decay and gravitational collapse, but transparent to CMB and also to the major part of the electromagnetic spectrum.

Conclusion

In this paper we have taken some of the approximations derived for the dipole matrix elements for electromagnetic decay and used them to calculate decay times for the gravitational eigenstates of potential wells surrounding central masses of various sizes. It is significant that when the central potential well is deep and the quantum numbers large, that some of the eigenstates can have lifetimes that far exceed the age of the universe. These states correspond to the high angular momentum, angularly delocalized eigenstates that exist at large radii. The low values for spontaneous decay rates also correspond to low rates for stimulated emission and absorption. These stimulated rates remain low even for transitions from low \( p \) states that involve large quantum jumps because of the rapidly decreasing overlap integrals. This means that high \( n \) low \( p \) eigenstates are gravitationally stable and transparent to most of the electromagnetic spectrum. Hence these states may be considered as candidates for dark matter.

References