The Properties of Eigenstates in Weak Gravity

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Abstract
Schrödinger’s equation can be applied to the weak regions of deep gravitational wells provided the eigenstates do not extend into regions of strong gravity where relativistic approach would be required. Using this approach we find the existence of certain stable, high angular momentum eigenstate solutions that have no classical analogue, and unusual properties radically different to those of traditionally localized matter or orbiting particles. States which are realistically bound necessarily involve large quantum numbers, the study of which requires special mathematical techniques. We present some the results from the mathematical techniques which we have derived to deal with large quantum valued states, and investigate some of the more interesting properties of these states, which include extremely long lifetimes and low interaction rates with photons and localized particles. In particular we determine the broad structure, position and extent of the relevant eigenstates, and estimates of the values of the overlap integrals that ultimately lead to the various interaction rates. We also examine some of the astrophysical consequences of the existence of gravitational eigenstates that have these unusual properties.

Introduction
The bound stationary states of electrons in atoms have been studied extensively for many years but little has been done to investigate the properties of neutral or charged particles occupying the hypothetical stationary states in gravitational potential wells. This is important not only because quantum theory predicts the existence of such states but also because recent experimental work (Nesvizhevsky et al 2003) has indeed demonstrated their physical reality. It is interesting therefore to speculate on whether relatively pure stationary gravitational eigenstates to exist naturally elsewhere in the universe. If such states were to exist, it would clearly be important to have theoretical information on their expected properties.

The gravitational form of Schrödinger’s equation in weak gravity may be written as

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi - \frac{G\mu M}{r} \psi = i\hbar \frac{\partial \psi}{\partial t}$$

where $M$ is the central mass, $\mu$ is the reduced mass and the other symbols have their normal meanings. The solutions are analogous to those of the hydrogen atom and may be immediately written down as

$$u_{n,l,m}(r,t) = R_{n,l}(r)Y_{l,m}(\theta,\phi)$$

where $Y_{l,m}(\theta,\phi)$ are the normalized spherical harmonics and

$$R_{n,l}(r) = N_{n,l} \left(\frac{2r}{\hbar_0}\right)^l \exp\left(-\frac{r}{\hbar_0}\right) I_{n-l-1}^{2l+1}\left(\frac{2r}{\hbar_0}\right)$$

In equation (3) $N_{n,l} = \left[\frac{2}{\hbar_0}\right]^3 \frac{(n-l-1)!}{2n(n+l)!}^\frac{1}{2}$ is a normalizing constant and $\hbar_0 = \frac{\hbar^2}{G\mu m_p M}$ is a scale parameter and

$$I_{n-l-1}^{2l+1}\left(\frac{2r}{\hbar_0}\right) = (n+l)! \sum_{k=0}^{n-l-1} \frac{(-1)^k + 2l}{(2l+1+k)!k!}$$

are the generalized Laguerre polynomials in their standard form.

It has been shown previously (Ernest 2005, 2008) that the decay rates of particles in some of the high-$n$, high-$l$ quantum number solutions to (1) show some very unusual and unexpected properties compared to their atomic counterparts. These particular solutions can have exceptionally long lifetimes (in some cases many times the age of the universe) and other properties that are significantly different from their atomic counterparts, even though the solutions for the simplest gravitational potentials such as that described in (1) are essentially the same as the atomic case. The difference in properties comes about as a result of the fact that when $n$ and $l$ are very large, well bound
eigenstates are predicted for large mass central potentials by the gravitational form of Schrödinger’s equation. In the atomic case such large values of \( n \) would never be considered since they would correspond to energy states which were too close the continuum to be realistic. In the gravitational case large \( n, l \) eigenstates are viable states for consideration; they can be well bound and possess the same levels of angular momentum and energy as orbiting particles yet exist as stationary state solutions. These states are completely different however, from an orbiting particle with the equivalent energy and momentum since they are totally delocalized in the azimuthal direction and also have significantly large spread in the polar and radial directions, depending on the particular value of \( n, m, \) and \( l \). Although these states are often in the macroscopic domain, there is no evidence from quantum mechanics to deny their existence. Decoherence, which limits the observation of large-mass interference effects, affects only the phase of an eigenstate (Reynaud 2001) and does not necessarily lead to macroscopic localisation. Very little has been done to study the properties of the high quantum number eigenfunctions and many of their unusual properties require approximate methods because the explicit expressions for them involve an intractable number of terms making exact analysis impossible.

The interaction cross-section for any reaction is determined by an appropriate overlap integral involving the initial and final states of the system and the Hamiltonian describing the interaction potential. The value of the overlap integral is dependent in part on the similarity of the two states. We are concerned here with the degree to which size and shape of the individual eigenfunctions influence the overlap integral. When two eigenfunctions show little or no overlap for example the overlap integral cannot be large and the interaction cross-section therefore small for that particular change in state. Additionally there are structural factors of the high-\( n \) eigenstates other than the degree of overlap that can cause an overlap integral to be small irrespective of the nature of the interaction potential.

It was shown (Ernest 2008) that the angular components of any gravitational eigenstate always overlap in space and that the radial part of the overlap integral can often be responsible for its weakly interacting nature. It was also shown that the angular components of the electromagnetic dipole matrix element have limited size and converge to small values (\(< 1\)) for large \( l \) irrespective of the value of \( n \). This is also expected to be the case for other types of matrix elements such as those for electromagnetic quadrupole and multipole transitions and particle interactions.

The radial component of any matrix element can have a very significant influence on the weakly interactive nature of state transitions. This component can be zero, for example, if there is no overlap, or alternatively, very large if the interaction Hamiltonian contains factors of \( r \). It turns out that the structural features of the radial component eigenfunctions can often have much more influence on the value of the overlap integral than the form of the Hamiltonian itself, and so it is worthwhile to investigate how the position, radial width and other internal structural features such as spatial oscillation frequency vary with the quantum parameters \( n \) and \( l \). The earlier papers (Ernest 2008) developed approximate methods for determining the position, and radial width of the radial eigenfunction components and also their spatial oscillation frequencies. In this paper we apply these approximations to determine specific values for the position, extent and spatial oscillation frequencies for interesting cases.

**Physical Properties of the Eigenstates**

The approximation developed previously (Ernest 2005, 2008) involved the empirical observation that the spatial positions of the zeroes of the radial eigenfunctions follow an approximate square root dependence

\[
\sqrt{\gamma_i} = \sqrt{\gamma_1} + (i-1)\delta
\]  

(5)

where \( \delta \) is a spacing parameter and \( \gamma_i \) is the lowest zero in the equation

\[
\Upsilon(\gamma) = \sum_{k=0}^{p} (-1)^{k-p+1}\gamma^k \frac{(p-1)!(2n-p)!}{(p-1-k)!(2n-2p+k+1)!k!}
\]

(6)

where \( p = n - l \) defines a new quantum variable, and the \( \gamma_i \) relate to the positions \( r_i \) of the zeroes of the radial eigenfunction via the equation \( \gamma_i = 2r_i/\hbar_0 \).

The empirical relation above (which is a better approximation when \( p \) is small relative to \( n \)) means that a plot of the square root of the positions of the zeroes against zero number gives a straight line which enables all the zeroes to be obtained once the parameters \( \gamma_1 \) and \( \delta \) are known. \( \gamma_1 \) and \( \delta \) are determined by using (5) and (6) together with simple polynomial theory to give a quartic equation in \( \gamma_1 \) after eliminating \( \delta \). Solutions to this quartic equation may be analytically obtained to give both \( \gamma_1 \) and \( \delta \). The zeroes obtained using these parameters and equation (5) are the as the zeroes of the radial eigenfunction. Knowledge of the zeroes of the radial eigenfunctions enables an
estimate of the radial width and position of the radial eigenfunction to be obtained. This is useful because it enables us to establish the relative overlap of the radial eigenfunctions in any overlap integral. The effectiveness of lack of overlap is enhanced by the fact that the size of any radial eigenfunction falls to an extremely low value outside it “range”. For the purposes of the present work we have taken the range of any particular radial function to be the difference between its first and last zero. This introduces an error which is of the order of twice zero spacing. However, for large \( n >> p >> 1 \) this is an insignificant fraction of the width and if two eigenfunctions are separated by a distance of more than 10 to 100 times their zero spacing any overlap integral is effectively zero (Ernest, 2008).

We show in fig. 1 plots of the position of the first and last zero of the radial eigenfunctions as a function of \( p \) value for three different values of \( n \). From this figure the approximate width of any eigenstate can be obtained from the spacing from the respective upper and lower curves, and the average position as the midpoint of this spacing. As expected the average position of the wave function increases with increasing \( n \) and the width of the eigenfunction increases with increasing \( p \). From this graph it is possible to determine which pairs of eigenstates exhibit potential for a non zero overlap integral. For example, for two states whose \( n = 9 \times 10^{33} \) and \( 9.5 \times 10^{33} \) there is no overlap between the states until \( p \) is of the order of \( 2 \times 10^{31} \). In particular the very low \( p \) eigenstates share no overlap with any other low \( p \) eigenstates and overlap is only possible when \( p \) approaches \( n \). In this case, however, two additional factors become important in limiting the value of any interaction cross-section. The first concerns the fact that as \( p \rightarrow n \), \( l \rightarrow 0 \), which corresponds to an eigenfunction of large radial spread (see fig. 1). This means that one of the eigenfunctions will have a very low amplitude over the effective range of the other (and over the practical range of the overlap integral) because of the normalisation amplitude. An estimate of the degree to which the amplitude decreases with increasing \( p \) is shown in figure 2. This severely limits the size of the integral and hence the interaction cross-section. An even more significant factor in limiting the overlap integral is the observation that as \( p \) increases the spatial oscillation frequency becomes very high (see fig. 3) and, for large \( n \), the oscillatory form of the eigenfunction becomes very uniform. This means that the overlap integral in situations where the two eigenfunctions have high \( n \) values but very different \( p \) values will be extremely small. This is because the integral is comprised of two functions which have vastly different spatial oscillatory behaviour: the low \( p \) eigenfunction is slowly but uniformly oscillating in the radial direction while the high \( p \) eigenfunction is oscillatory much more rapidly and uniformly under the profile of the former. For example if we consider a \( p = 1 \) state for \( n = 10^{34} \) the radial spread is approximately \( 4 \times 10^{7} \) m. For a change in \( n \) of \( 4 \times 10^{7} \) m the spatial oscillation frequency of the second wave function is such that there are over \( 10^{10} \) uniform oscillations of one eigenfunction within one oscillation of the other in addition the amplitude. The effect of this combination of decreased amplitude and increased spatial oscillation frequency can be seen by direct calculation of the approximation to the overlap integral for transitions involving one \( p = 1 \) eigenstate when \( n >> p \) is given by,

\[
I_R = \frac{2^{n_f^2} \hbar_0 (p_i + n_f)^{n_f + 2}}{\sqrt{e}} \left( \frac{n_f^2}{\pi p_i^{n_f + 2}} \right) ^\frac{1}{4} \left( \frac{2n_f + p_i + 1}{2n_f + p_i} \right) ^\frac{4n_f + 2p_i + 1}{4} \left( \frac{2n_f + p_i}{2n_f + p_i} \right) ^\frac{2n_f + p_i + 2}{2n_f + p_i}
\]

(7)
where $p_i$ is the initial $p$ value $n_f$ is the final $n$ value.

This equation gives a value for the radial component of the overlap integral $I_R$ of $\sim 10^{-60} \Omega$ and demonstrates the effectiveness of the combined reduction in amplitude and increase in spatial oscillation frequency in nullifying the value of the overlap integral. For overlap integrals where both eigenstates have large $p$ the situation is quite different. The practical limits of the integral extend over a much larger range and depending on their oscillatory behaviour, the overlap integral can be such that there can be significant transitions between states.

The significance of the prediction of the existence of these weakly interacting gravitational eigenstates by quantum theory may have some important consequences for astrophysics. If traditional baryonic particles such as electrons and protons were placed into such states they would become essentially invisible and weakly interacting making them detectable only by their gravitational influence. It was shown in (Ernest 2005) that a uniform ensemble of protons and electrons in such eigenstates in the halo of the galaxy would have potential energies that were dominated by their gravitational rather than their electromagnetic properties yet would not significantly undergo atomic recombination. It was also shown in (Ernest 2005) that most of the astrophysical observations concerning dark matter could be understood in terms of the existence of stable eigenstates yet avoided some of the difficulties with traditional “all baryonic” dark matter models, particularly the arguments that arise from Big Bang Nucleosynthesis theory. Additionally the quantum approach to dark matter predicts the expected visible to total matter ratio, and is consistent with $1/r^2$ density profiles and halo sizes.

Conclusion
The present results are of interest as they show that, unlike the atomic case there exists a set of eigenstates within the wells of sufficiently deep gravitational potentials that have very low interaction rates with all other members of the eigenstate ensemble. These states when populated by traditional baryonic particles will exhibit different cross-sections to that expected from a group of particles with a Maxwellian distribution. In particular the well bound, high-$n$ low-$p$ eigenstates show extremely long lifetimes (in some situations $>10^{25}$ s) and an inherent inability to gravitationally collapse. Additionally, these long lifetimes mean that these states will be also transparent to electromagnetic radiation (because of the corresponding Einstein $B$ coefficient (Ernest 2005)). Furthermore, since the weakly interacting nature of the low-$p$ states is due to the structure of the eigenstates themselves rather than interaction Hamiltonian other types of interactions are also likely to be small, the value being determined by the number of available transition channels and the selection rules for the process.

References