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Abstract: A graceful labelling of an undirected graph *G* with *n* edges is a one-to-one function from the set of vertices of *G* to the set *(0, 1, 2,…, n)* such that the induced edge labels are all distinct. An induced edge label is the absolute value of the difference between the two end-vertex labels. The Graceful Tree Conjecture states that all trees have a graceful labelling. In this survey we present known results towards proving the Graceful Tree Conjecture.

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Towards the Graceful Tree Conjecture: A Survey

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Abstract

A graceful labelling of an undirected graph G with n edges is a one-to-one function from the set of vertices of G to the set $\{0, 1, 2, \ldots, n\}$ such that the induced edge labels are all distinct. An induced edge label is the absolute value of the difference between the two end-vertex labels. The Graceful Tree Conjecture states that all trees have a graceful labelling. In this survey we present known results towards proving the Graceful Tree Conjecture.

1 Introduction.

A graceful labelling of an undirected graph G with n edges is a one-to-one function from the set of vertices of G to the set $\{0, 1, 2, \ldots, n\}$ such that the induced edge labels are all distinct. An induced edge label is the absolute value of the difference between the two end-vertex labels. This labelling was originally introduced in 1967 by Rosa who also showed that the existence of a graceful labelling of a given graph G with n edges is a sufficient condition for the existence of a cyclic decomposition of a complete graph of order $2n+1$ into sub-graphs isomorphic to G. The famous Graceful Tree Conjecture (also known as Ringel-Kotzig, or Rosa's, or even Ringel-Kotzig-Rosa conjecture) says that all trees have a graceful labelling.

In this survey we give the state of the progress towards this 35 years old conjecture, fondly called a 'disease' of graph theory. In the next section we present some early results, and in Section 3 we give an overview of some classes of trees known to be graceful. Section 4 is devoted to the relaxed graceful labellings and Section 5 presents some open problems. Concluding remarks are given in Section 6.

2 The early results.

In his 1967 paper [26], Rosa presented four hierarchically related labellings of graphs, which he named α, β, σ and ρ-valuations. Let G be a simple graph with m vertices and n edges.

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A valuation O_G, is a one-to-one mapping of the vertex set of G into the set of non-negative integers. Let s be an edge between vertices v_i and v_j and let a_i and a_j be the labels of v_i and v_j, respectively. Then $b_s = |a_i - a_j|$ is referred to as an induced label of edge s. Let V_{O_G} be the set of vertex labels and H_{O_G} be the set of induced edge labels in a valuation O_G of the graph G. Rosa [26] defines the following conditions on a valuation O_G of a graph G:

(a) $V_{O_G} \subset \{0, 1, ..., n\}$;

(b) $V_{O_G} \subset \{0, 1, ..., 2n\}$;

(c) $H_{O_G} \equiv \{1, 2, ..., n\}$;

(d) $H_{O_G} \equiv \{x_1, x_2, ..., x_n\}$, where $x_i = i$ or $x_i = 2n + 1 - i$,

(e) there exists $x, x \in \{0, 1, ..., n\}$, such that for an arbitrary edge (v_i, v_j) of the graph G either $a_i \leq x < a_j$ or $a_j \leq x < a_i$ holds.

A valuation satisfying the conditions

$(a), (c), (e)$ is called an α-valuation,

$(a), (c)$ is called a β-valuation,

$(b), (c)$ is called a σ-valuation,

$(b), (d)$ is called a ρ-valuation.

In 1976 Sheppard [29] used the term balanced labelling to denote α-labelling. Also, β-valuation has been called graceful numbering by Golomb [17] in 1972, and proper labelling by Sheppard [29] in 1976. However, it is most commonly called graceful labelling, and in this paper we use the terms β-valuation and graceful labelling interchangeably. Refer to Figure 1 for an example of an α-valuation, and to Figure 2 and Figure 3 for examples of β-valuations.

From above definitions of the four valuations, it follows that they are hierarchical in the sequence $\alpha, \beta, \sigma, \rho$-valuation, in the sense that each valuation is a special case of succeeding valuation in the sequence. It is worth noting that if a graph G has an α-valuation, then it is a bipartite graph. Furthermore, if G is a complete bipartite graph, there exists an α-valuation of G [26](see Figure 1).

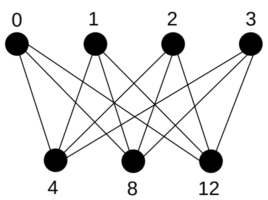

Figure 1: An example of an α-valuation

We now discuss some of the results which Rosa presented in [26]. He proved that if G is an Eulerian graph with n edges, where $n \equiv 1$ or 2 (mod 4), then there exists no β-valuation for G. Moreover, if the graph is a n-gon, an α-valuation exists if and only if $n \equiv 0$ (mod

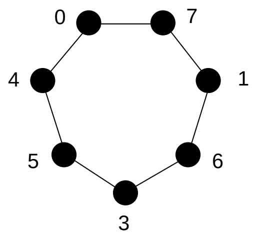

Figure 2: An example of a β-valuation

4), while a β-valuation exists whenever $n \equiv 0$ or 3 (mod 4). For an example of β-valuation on a 7-gon, see Figure 2.

In his original paper, Rosa [26] showed that caterpillars (Figure 3) and paths both have an α-valuation. Moreover, he showed that for an arbitrary n and an arbitrary vertex v of a path P_n with n edges [26, 27]:

a) there exists a β-valuation of P_n such that the label for v is 0;

b) there exists an α-valuation of P_n such that label for v is 0, if and only if v is not the central vertex of P_4.

However, it is worth noting that not all trees have an α-valuation. A *base* of a tree is a tree obtained by removing all the end vertices of the original tree. A tree T belongs to the class $\zeta(2,4)$ if the diameter of the tree is 4 and the base of its base is a tree with a unique vertex, but the base of T is not a path. No tree belonging to $\zeta(2,4)$ has an α-valuation [26, 15].

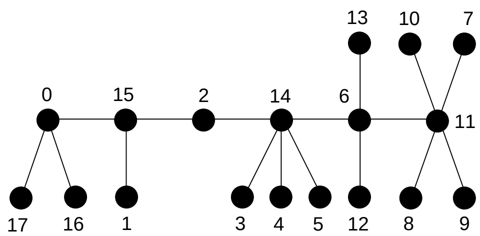

Figure 3: A graceful labelling on a caterpillar

Rosa [26] established a connection between the graph valuations and the well known Ringel's conjecture. In 1963 Ringel [25] conjectured that the complete graph K_{2n+1} can be decomposed into $2n+1$ subgraphs, which are all isomorphic to a given tree with n edges. Then Kotzig [19] conjectured that the complete graph on $2n+1$ vertices can be *cyclically* decomposed into $2n+1$ subgraphs which are all isomorphic to a given tree with n edges. Rosa [26] showed the connection between the ρ-valuation and the cyclic decompositions of the complete graph into isomorphic sub-graphs. Rosa proved that a cyclic decomposition of the complete graph K_{2n+1} into subgraphs isomorphic to a given graph G with n edges exists if and only if there exists a ρ-valuation of the graph G. In other words, Rosa proved that Kotzig's conjecture holds if and only if all trees have a ρ-valuation. The conjecture that every tree has a graceful labelling was first mentioned in [26]. This conjecture is known as the Graceful Tree Conjecture, and is also known as Ringel-Kotzig, Rosa's or Ringel-Kotzig-Rosa conjecture.

Huang, Kotzig and Rosa in [15] presented a series of trees that have no α-labelling, yet they admit a β-labelling. The authors suggested that an inductive proof of the Graceful Tree Conjecture would somehow have to combine trees with α-labelling and β-labelling. Let T_1 and T_2 to be two trees with mutually disjoint vertex-sets. Further let T_1 have an α-labelling O_{G_1} such that the label 0 is at some vertex $u \in V(T_1)$, and T_2 has a β-labelling O_{G_2} such that the label 0 is at vertex $v \in V(T_2)$. Then there exists a β-labelling of the tree $T = T_1 \bigcup T_2$ where $T_1 \bigcup T_2$ is obtained by identifying vertices u and v of trees T_1 and T_2.

By proving the following result Kotzig showed that almost all trees admit an α-labelling [19].

Theorem 1 *Let T be a tree, e an edge of T, $T_e(k)$ the tree which can be obtained from T if we replace e by a k-path with the same end vertices, and let $S_e(T)$ be the infinite set $S_e(T) = T_e(1), T_e(2), \ldots$. Then the number of trees in $S_e(T)$ without an α-valuation is finite for every edge e of T.*

On the other hand, an unpublished result by Erdos [16] states that most graphs are not graceful.

In the next section we present some classes of trees that are known to be graceful.

3 Further results towards the conjecture.

In 1998 Aldred and McKay [1] used computer search to prove that all trees on at most 27 vertices are graceful.

Although we know that paths can always be labelled gracefully [26], the number of graceful labellings of a path of length n is not known. A recent result by Aldred, Širáň and Širáň in 2003 [2] shows that this number grows asymptotically at least as fast as $(\frac{5}{3})^n$.

Caterpillars were shown to be graceful early on by Rosa [26]. They can be labelled using a similar strategy as for paths, as shown in Figure 3.

It was conjectured by Bermond in 1979 [3] that the class of graphs known as lobsters are graceful. A lobster is a tree whose base is a caterpillar. This conjecture is still open, with only a few limited cases being solved ([23], [31], [11]). A firecracker is a special class of lobster and can be seen as a collection of stars where one end vertex from each star is chosen and they are all connected in a path. Chen et al. [11] proved in 1997 that all firecrackers are graceful. See the Figure 4 below for an example labelling.

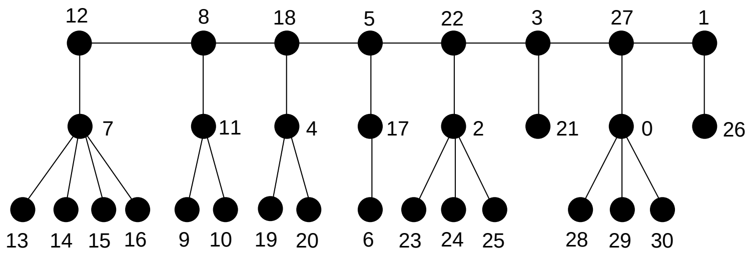

Figure 4: An example graceful labelling for a firecracker

A banana tree consists of a collection of stars and the vertex v, where one end vertex from each star is joined to the vertex v (see Figure 5). Chen, Lü and Yeh [11] have conjectured

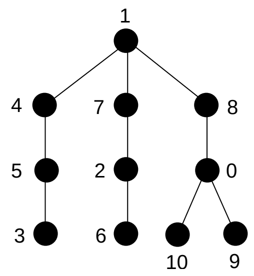

Figure 5: An example graceful labelling for a banana tree

that all banana trees are graceful, and indeed some classes of banana trees been shown to be graceful [5].

All trees with at most four end vertices are graceful [26]. Zhao [32] proved that all trees with diameter at most four are graceful. This result was extended in 2001 by Hrnčiar and Haviar [14] to show that all trees with diameter at most five are graceful.

An olive tree is a collection of i paths joined in a vertex, where path i has length i. Olive trees were shown to be graceful in 1978 by Pastel and Raynaud [24], (see Figure 6).

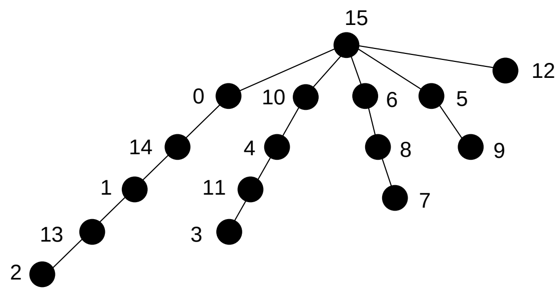

Figure 6: An olive tree with graceful labelling

A symmetrical tree is a rooted tree where all the vertices at the same distance from the root have the same degree. All symmetrical trees are graceful, a result by Bermond and Sotteau in 1975 [4], (see Figure 7).

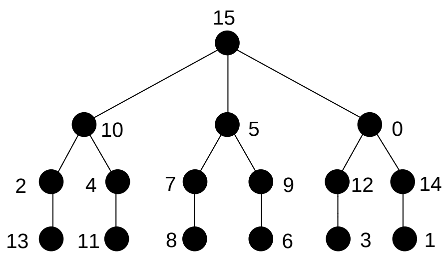

Figure 7: An example of graceful labelling for a symmetrical tree

Methods for combining graceful trees to obtain new graceful trees were described by Stanton and Zarnke in 1973 [30] and by Koh, Rogers and Tan in 1979 [18].

In 1998 Burzio and Ferrarese [9] showed that all trees obtained from graceful trees by replacing every edge with a path of length two, are themselves graceful.

In 2002, Hegde and Shetty [13] proved that a class of trees that can be transformed into a path by a sequence of "elementary parallel transformations" are graceful.

Morgan et al [21, 22] explored the relationship between Skolem and Hooked-Skolem sequences and graceful labellings.

Broersma and Hoede [8] defined the concept of *strongly graceful labellings* for trees with perfect matching. Let T be a tree with m vertices and a perfect matching M. A gracefull labelling O_T of T such that for every edge $\{uv\} \in M$, $a_u + a_v = m - 1$, where a_u and a_v are the labels of u and v respectively, is called a strongly graceful labelling of T. A tree T is said to be strongly graceful if and only if there exists a stongly graceful labelling of T. Broersma and Hoede [8] showed that the following conjecture is equivalent to the Graceful Tree Conjecture.

Conjecture 1 *Every tree containing a perfect matching is strongly graceful.*

4 Relaxed graceful labelling.

Note that a labeling $f : V \to \{0, 1, \ldots, n\}$ of a graph $G = (V, E)$ with m vertices and n edges is graceful if it is an injection and the induced edge labeling $g : E \to \{1, 2, \ldots, n\}$ is a bijection.

Bussel [10] considered the different relaxations of graceful labellings and classified them into the following three categories:

1. Edge-relaxed graceful labelling where repeated edges are allowed. Thus g is any function from E to $\{1, 2, \ldots, n\}$ and not necessarily a bijection.
2. Vertex-relaxed graceful labelling where repeated vertex labels are allowed. Thus f is relaxed to be any function from V to $\{0, 1, \ldots, n\}$ and not necessarily an injection.
3. Range-relaxed graceful labelling where there are more labels available for vertices than $n + 1$, and induced edge labels need only to be distinct and not necessarily in the range $[1, n]$. More formally, in range relaxed labeling f is an injection from V to $\{0, 1, \ldots, n'\}$ where $n' > n$ and g is an injection from E to $\{1, 2, \ldots, n'\}$.

Most attention has been paid to the edge-relaxed graceful labelling. In order to present these results, we first need to introduce the so-called *bipartite labeling*. A labeling f is bipartite if there exists k such that for every edge uv we have either

$$f(u) \leq k < f(v) \text{ or } f(v) \leq k < f(u).$$

A bipartite graceful labelling is in fact an α-labeling as defined by Rosa [26]. Let $T = (V, E)$ be a tree with n edges. The α-size $\alpha(T)$ of the tree T is the maximum number of distinct induced edge labels taken over all bipartite labellings of T [28]. Similarly, the β-size or the *grace size* $gs(T)$ of the tree T is the maximum number of distinct induced edge labels taken over all labellings of T. If the "grace size" of a tree T is equal to the number of edges in T, then T is graceful. The grace size is in a sense opposite to the "bandsize" of a graph G [12] which is the minimum number of distinct edge labels taken over all labellings of the graph G. This problem is closely related to the "bandwidth" of a graph where the goal is to label the vertices of the graph in such a way that the maximal induced edge label, i.e. the bandwidth, is minimized. Mohar [20] used spectral graph theory to construct a close to optimal solution to the bandwidth problem. It appears that a similar method can be used to prove that some classes of graphs do not have a graceful labelling.

Rosa and Širáň [28] investigated the asymptotic behavior of α-size of trees. $\alpha(m)$ is the minimum of $\alpha(T)$ over all trees with m vertices. They showed: $\frac{5m}{7} \leq \alpha(m) \leq \frac{(5m+4)}{6}$ for all $m \geq 4$. Consequently the grace size $gs(T)$ of any tree with m vertices satisfies $gs(T) \geq \frac{5m}{7}$.

Bonnington and Širáň [6] investigated the α-size of trees with maximum degree three. They considered $\alpha_3(m)$ to be the smallest α-size among all such trees with m vertices. They proved that $\alpha_3(m) \geq \frac{5m}{6}$ for all $m \geq 12$. Brankovic, Rosa and Širáň [7] improved the lower bound for the α-size of trees with maximum degree three to: $\alpha_3(m) \geq \lfloor \frac{6m}{7} \rfloor - 1$.

Bussel [10] investigated vertex-relaxed graceful labelling. He proved that every tree T on m vertices has a vertex-relaxed graceful labelling such that the number of distinct vertex labels is strictly greater than $\frac{m}{2}$. He also investigated range-relaxed graceful labelling, and proved that every tree T on n edges has a range-relaxed graceful labelling f with vertex labels in the range $0, \ldots, 2n-$diameter(T).

5 Open problems.

There are so many ways to restrict the Graceful Tree Conjecture in order to obtain a problem that is potentially easier to solve than the Conjecture. We provide a list of 5 such problems in what we believe to be in order of increasing difficulty.

- Paths of arbitrary lengths connected by a vertex.
- All trees with 5 leaves.
- The full class of lobsters.
- Trees of diameter 6 are graceful.
- All trees with maximum degree 3.

6 Conclusion.

The Graceful Tree Conjecture was intially interesting mostly because of its connection to Ringel's Conjecture, but soon became famous in its own right. Despite efforts of many researchers, only limited progress has been made over the last few decades. To date, only some very restricted classes of trees have been shown to be graceful. Although some progress has been made on the relaxed labellings, we still do not seem to be much closer to a solution. Nevertheless, the faith in the Conjecture is so strong that if a tree without a graceful labelling were indeed found, then it probably would not be considered a tree.

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