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International Perspectives on Learning and Teaching Mathematics

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Contents

Improving the Teaching and Learning of Space Mathematics

KAY OWENS

This chapter presents results from two studies implementing a framework of space mathematics that emphasised investigating and visualising together with describing and classifying. Two key ideas were (i) part-whole relationships and (ii) orientation and motion. A number of schools participated in the project and many lessons were developed. While early primary school classes were used, the lessons are applicable to higher levels of primary school too. Teacher knowledge and their feedback on the lessons and program were evaluated. Some classroom scenarios together with the lesson plans will illustrate the value of teachers using questioning and concrete material. When teachers were clear about the purpose for lessons they could facilitate deeper understanding by students. Videotapes helped teachers understand the framework. Teachers were also provided with some task-based interview schedules to assist them in assessing their students' knowledge before they started. They worked with facilitators who worked in the classrooms and modelled good teaching practice.

The framework for the *Count Me into Space* (CMIS) program

Since the successful introduction of the *Count Me In Too* program on number in the early primary schools of New South Wales (aged 5 to 8), the Professional Support and Curriculum Unit (led by Peter Gould and Diane McPhail) and I developed a similar research-based program for space mathematics called *Count Me Into Space* (CMIS). Other Australian researchers, especially Michael Mitchelmore, have had some input. We developed a framework that was intended to capture the wealth of diverse research in this area and scaffold teachers' planning of learning experiences.

The number project used terms like emergent, perceptual and figurative stages of development. For the CMIS project, these terms were built upon and given meaning in terms of the spatial abilities and visual imagery literature. Like the number project, the CMIS project assessed young students' spatial learning using individual task-based interviews. The students' responses were indicative of different strategies which were named emerging strategies, perceptual strategies, pictorial imagery strategies, pattern and dynamic imagery strategies, and efficient strategies. Each of these strategies reflect the research literature.

Emerging strategies

Pirie and Kieran (1992) refer to this beginning as *primitive knowing* on which *image making* is developed. Van Hiele (1986) refers to *intuition*. By four and a half years, children are able to distinguish wholes and parts of simple designs such as plus or cross signs (Feeney & Stiles, 1996). Both Macmillan (1998) and Rogers (1999) have shown that young children develop mathematical knowledge through play and they discuss qualitative and quantitative spatial ideas. The children use (a) position language of degree, e.g., halfway, near; (b) shape and line names and classification characteristics, e.g. "like a window", (c) enjoyment at seeing and making spatial patterns, (d) turns and corners and (e) patterns of area. Rogers illustrated how children use (a) visualisation in their preconceived plan, projection and refinement; (b) experimentation in concrete problem solving for balanced, symmetrical, and aesthetic structures, and (c) application in making new structures and deciding their purpose as both real and imagined products. Interaction between children but also adults' modelling, acceptance, positive responses, and questioning helped students' confidence, cooperation, and expression of mathematical ideas and purposes.

In another study of students selecting and covering different shapes, untrained children in their first year of school solved the tasks by persistence rather than using more efficient or varied strategies. Nevertheless they tended to recognise shapes which would not lead to a solution, and to re-position pieces (Mansfield & Scott, 1990).

CMIS summarises emerging strategies as students *beginning to* attend purposefully to aspects of spatial experiences, to manipulate and explore shapes and space, to select shapes like ones shown or named, and to associate words with shapes and positions.

Perceptual strategies

These beginning or emerging strategies need to be developed into more efficient strategies. When children first begin to reason geometrically, they use direct or indirect resemblance and real world referents. Later they reason by attributes and then by properties (Fox, 2000; Lehrer, Fennema, Carpenter, & Ansell, 1994). Attending to and disembedding features of shapes becomes a critical skill when students learn about 2D and 3D shapes (Owens & Clements, 1998). Flavell (1977) commented that attentional processes become increasingly inter woven with other cognitive processes such as memory, learning, and intelligence. Attention is attracted by perceptually outstanding features such as size and special form (Flavell, 1977), pictures (Everett, 1999), number of items, and the inherent interest of the items for a child (Vurpillot, 1976). In addition, language and experiences of alternatives for a space concept are important.

CMIS summarises students using perceptual strategies as attending to spatial features and beginning to make comparisons, relying on what they can see or do.

Pictorial imagery strategies

Students' initial images of concepts are generally static. These prototypical images may have certain features that the students incorrectly associate with a concept. For example, a student might have a fixed image of a rectangle with a horizontal base that is roughly two squares. Long thin rectangles, obliquely placed rectangles and squares would be discounted by this limited image. However the initial image can be a good place to start developing rich conceptual imagery and solve problems. Metonymy, the use of a part of a concept for the whole concept, can also assist the student with locating schema in the mind and selectively attending to aspects of a problem-solving situation. Since "imaging involves three activities: constructing an image, re-presenting the image, and transforming the image" (Wheatley, 1998, p. 66), the challenge for teachers is to move students from a construction of a rigid mental picture which can limit reasoning. Attention to and discussion of key defining features of shapes and pictorial imagery strategies are encouraged by activities that require students to predict and create.

CMIS summarises students using pictorial imagery strategies as developing mental images associated with concepts with increasing use of standard language.

Pattern and dynamic imagery strategies

Students in early primary school begin to reason about shapes by considering certain features of the shapes as well as using their prototypical images (Clements, Swaminathan, Hannibal, & Sarama, 1999). Presmeg (1986) first discussed the diversity of imagery that high school students use in remembering and using formulae. Owens (1996; Owens & Clements, 1998) found that primary school students were using similar dynamic and pattern imagery strategies while solving spatial problems.

The CMIS project developed activities that would allow students to experience shapes in different orientations, changes, positions, and sizes so that they could develop and discuss their imagery. Shapes made from elastic, drawn on stretchy material, or in a computer drawing package can be stretched to give a good sense of the diversity of shapes that are all called by the same name. Stretching shapes also helps students to see how shapes are related by the number and type of parts. By forming a rectangle and pulling a side so sides remain parallel and the angles at ninety degrees, a full range of rectangles including the square can be made. Forming a rhombus with diagonals and squashing it allows students to see the square is included and the diagonals always remain at right-angles. These explorations develop *dynamic imagery*. A student whose imagery is dynamic can imagine making many different triangles by pulling a corner. They can reason about lengths of sides. Discussion about the changes and the properties that vary or stay the same will help to develop students' conceptual understandings. Tessellating shapes and covering areas helps students to develop pattern imagery.

CMIS summarises students using pattern and dynamic imagery strategies as using pattern and movement in their mental imagery and developing conceptual relationships.

Efficient strategies

Spatial thinking may be considered as spatial abilities or visual processing. It includes transforming shapes, inspecting and disembedding parts within shape configurations and visual scanning (Eliot, 1987; Kosslyn, 1981). It also involves spatial conceptualising and the interaction of visual imagery with these concepts (Piaget & Inhelder, 1956). "Image schemata can facilitate mathematical reasoning because their internal structure can be extended figuratively to develop understanding of formal relations among concepts and propositions" (English, 1997, p. 10). Images may need to be re-presented when used. It is important that the imagery be dynamic if it is to assist with reasoning. That is, the imagery is mentally pliable, able to be rotated, stretched, shrunk and turned around. While spatial thinking is evident in the earlier strategies, its flexibility and efficiency increases with conceptual and visual reasoning. Students talking about their imagery and reasoning and developing concepts will strengthen the efficiency of strategies.

CMIS summarises students using efficient strategies as students selecting from a range of spatial strategies that are appropriate for a particular problem or concept. They efficiently use imagery, classification, part-whole relationships, and orientation and movement. They may be using procedural imagery by which they hold a series of procedures in their imagination. They are using geometric knowledge.

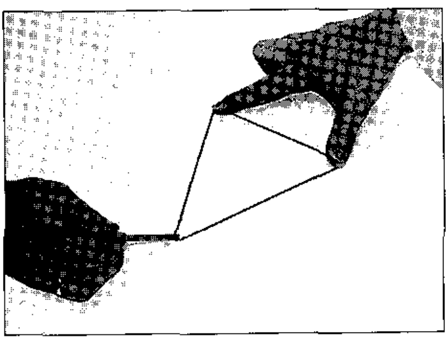
The probable sequence of strategy developments in students' learning can be about part-whole relationships or orientation and motion. Part-whole relationships refer to a shape's properties and classification. Orientation and motion refers to transformations of shapes. Two types of learning approaches (a) investigating and visualising, and (b) describing and classifying are emphasised. In this short paper, I will concentrate on spatial thinking involving orientation and motion. Table 1 summarises the framework for orientation and motion.

Table 1. *A framework for space mathematics – Orientation and motion.*

	Learning approach	
Type of Strategi	Investigating and visualising	Describing and classifying
	The student:	The student:
Emerging Strategies	recognises shapes that match the *child's fixed image(s)*	uses a shape word for a fixed image
Perceptual Strategies	recognises shapes in different orientations and proportions, checking by physical manipulation	describes similarities and differences and processes of change as they use materials
Pictorial Imagery Strategie	generates a series of static images of shapes in a variety of orientations and with different features	discusses shapes, their parts, and simple actions when the 2D and 3D shapes are not present but recently seen
Pattern and Dynamic Imagery Strategies	predicts changes by *mentally* modifying shapes and their attributes using motion or pattern analysis represent patterns and relationships of change by modelling or drawing	*describes a number of changes that will* occur with one or more actions discusses patterns and movements associated with combinations of shapes and relationships between shapes
Efficient *Strategies*	selects effective strategies to make *changes needed to achieve a* planned product	describes effective use of properties of shapes to generate new shapes

The assessment tasks

Figure 1 displays one of the tasks for the interview and illustrates how the strategies might be expressed by a student. Teachers assess six to eight students in their class in order to appreciate the framework and to gain knowledge of their students' current strategies. The instructions are given to teachers.



Task 3

Place the 25 cm loop of string on the table and hold two points firm with your thumb and forefinger of the same hand, about 10 cm apart (the string needs to form a triangle when the remaining loop is pulled with the stick). Provide the student with the stick.

→

Questions	Framework strategies
Task 3 (i)	
– Use this stick to pull the string tight and make a triangle. – How would you describe the triangle you have made?	*Emerging strategies.* Moves the stick but does not make or recognise the triangle. *Working towards Perceptual strategies.* Makes a triangle quickly. Talks about three sides or three corners (points or angles) *If the strategy is not shown, stop*
Task 3 (ii) – Could you make other triangles? Show me. Tell me about them. *Ask the student to hold the stick still forming a triangle, preferably not the equilateral triangle.* – Point with your finger to where the stick will be to make this side shorter? *(point to the shorter side between the stick and your finger)* – What will happen to the other side?	*Perceptual strategies.* Confirm if the student makes another triangle. *Pictorial imagery strategies.* Makes several triangles and points to a part of the string where the stick could be placed to make the side shorter and says the other side will be longer. *If the strategy is not shown, stop*
Task 3 (iii) – Tell me more about making different triangles. *Give the student paper and a pencil. Draw a vertical line on the sheet of paper, similar to the piece of string you were holding.* *Put away the stick, and say* – Draw the various triangles you have made but always use this line as the side of the triangle. It is like the string I was holding.	*Pattern and dynamic imagery strategies.* Explains that a large range of triangles can be made, illustrating by sliding the stick, Points to a position outside the original circle in the previous task which is where the stick really would be if the side were shorter Draws at least three triangles using the provided line so triangles are overlapping. *Efficient strategies.* Explains that a continuous range of triangles can be made and the stick will trace out an arc. Draws many triangles using the provided line with the third point tracing out the arc.

Figure 1. *Task from the interview on orientation and motion.*

Symmetry is a major aspect of orientation and motion thinking. The selection of a wrong line of symmetry is resilient. In one study, even student teachers confused congruence and mirror image, and were hampered by the inclination of the line, especially if a visual alternative was dominant like the oblique side of a parallelogram (Leikin, Berman, & Zaslavzky, 2000). Thomas (1978) found students in grades 1 and 3 were less able than those in grade 6 to locate a point on the side of a triangle once it was rotated or flipped but the older students considered the vertices as well as the sides of the triangle. Results seemed to be affected by the strategy used to make the decision or some features of the task rather than conservation. Vurpillot (1976) explained that a horizontal reference line in spatial perception tasks encourages subjective preference for distinguishing a "top" and a "bottom" of a shape while a vertical reference line encourages recognition of symmetry. Shape orientation, shapes with rotational symmetry or asymmetry (e.g., *J*) (Perham, 1978), lack of familiarity, unexpected sizes of shapes, and distance of displacement (Schultz, 1978) make transformation tasks more difficult. Students are able to do translations more easily than rotations and diagonal reflections, and half-turn clockwise is more difficult than two reflections or counter-clockwise rotation (Perham,1978).

The learning experiences

The CMIS project developed activities that would allow students to experience shapes in different orientations, positions, and sizes as well as dynamic changes to shapes so that they could develop and discuss their imagery. In orientation and motion, early activities included construction play, wrapping up boxes and dollies, and stretching shapes. Next were perceptual activities like making shapes with string, threading holes in different patterns, finding from where photographs were taken, printing faces of shapes, making jigsaws, playing a game to select the matching pieces of cut shapes, block building from an isometric diagram, and making a simple house shape using origami folds.

In particular, the difficulties with symmetry were addressed by activities designed to encourage pictorial imagery strategies. These included memory of a design, paper folding, pattern block symmetry, making similar block arrangements, geoboard symmetry, shadow shapes, drawing the view of a fly on the ceiling, and making models from drawings. These were followed by activities especially designed for developing pattern and dynamic imagery strategies: faces and nets, pop-up cards, paper box folding, around and around (e.g., a model from 3D shapes is drawn and students predict the position of drawings), and making a cylinder and a cone from paper.

It was the encouragement of students making, drawing, and discussing that made a difference. The following snippet from an observed lesson illustrates how teachers discussed shapes with the students and used the activities as a source for discussion with the whole class and small groups.

T. *This time I'm going to fold it in half and I'm going to draw a different shape on it. Remember the shape has to start and finish on the fold. What shape will I draw?*
S. *Not a usual shape.*
T. *There's my shape. Who can predict what it will look like if I cut it with scissors around here and open it out?*
S. *It will look like the end of a ribbon.*
T. *Yes it does look a bit like the end of a ribbon. Can you tell me about the sides?*
S. *They go like that (student indicates the outline in the air using index finger)*
T. *How many sides are there?*
S. *Five.*
T. *How do you know there's five?*
S. *Because you can't count that (indicates the fold) because when you cut it there, it won't be there*
T. *How many sides will there be?*
S. *Eight*
T. *How do you know?*
S. *Because it's half* (four sides are visible).

(After this long discussion at the start of the lesson, the students are given scissors and squares and they discuss with their neighbours what they are going to do and then they try it out. During this time the teacher asks questions. At the end of the lesson, the students share again.)

T. *How did you fold it?*
S. *Vertically. Then I looked at the half on it. Then I traced it on the side. Then I imagined it. Then I cut it out.*
S. *I folded it horizontally. Then I thinked [sic] about it. Then I drew it. Then it didn't turn out symmetrical so I had to change it.*
Further questions and answers followed.

For each of the lessons that were prepared for teachers, questions and teaching points were provided.

Faces and nets

Students use boxes to predict and investigate the nets of 3 dimensional objects.
Purpose:
This activity involves 2D to 3D transformations. Prediction encourages visual imagery.
→

Count Me Into Space framework:

- Consolidating Perceptual Strategies
  Recognises shapes that match a set image
- Working towards Pictorial Imagery Strategies
  Generates images of shapes in a variety of orientations and with different features
  Discusses shapes, their parts and actions when the shape is not present

Materials: A variety of different shaped small boxes from thin cardboard, sticky tape, scissors, two sheets of A3 paper and pencils.

Activities	Teaching points and questions
Introduction (whole class)	
– Discuss the different shapes and names of the different boxes. – Discuss the different parts of the boxes– faces, edges, corners – *Select a rectangular prism and ask* students where they might cut to flatten it out.	– Do all the boxes have the same shaped faces? – What do we call this shape? Why? – What is the difference between an edge and a face? – What makes the boxes different from each other? – Where might I cut next? – Might I have cut it out a different way?
Activity (individual)	
– Students select a box to investigate. – Before students trace or cut around the box, the teacher asks them to predict what shape a piece of paper would have to be if it was going to be pasted onto the sides of the box. – Students draw their prediction on a piece of paper. – Students cut along some of the edges of the box so that it can be flattened out. – *On the other sheet, students trace* around the shape to draw its net. – Students fold the net to make the box	– Allow the students time to investigate their box before trying to draw the shape. – Did the students predict correctly? – Did the students draw the shape of the face of the box in similar proportions? – How many faces can you see in your drawing? Is this the same as the box? – How do you make an edge when folding up the box? (fold and rub crease)
Conclusion (whole class)	
– Students bring their drawings and their made boxes to the front. – Students compare and discuss their drawings – Discuss the shapes they folded up.	– Do all the drawings look the same? – Was your tracing the same as others in the class? Why/Why not? – Tell me about your drawing. – Tell me about how you folded up the box. Encourage students to use correct terminology.

The study

The first study of the CMIS project was conducted over two years in 2000–2001 and aimed at assessing students' spatial thinking and seeing whether a series of lessons had an impact on students' learning. The research questions were: (a) Are students developing investigating and visualising strategies and progressing through the strategies in comparison to students in matched non-intervention schools? (b) Have students in Year 3 developed better 3D spatial thinking skills in comparison to students in non-intervention schools? In particular, did low attaining students improve? In addition, teacher feedback was used to evaluate teachers' efficacy for teaching space with this program and to evaluate and improve the program.

Methodology

Five schools in three school districts in Sydney, Australia, participated in the project. Students in Kindergarten to Year 3 were involved. Only data for Year 2 students who participated in lessons that emphasised orientation and motion are presented. Intervention schools were matched with a non-intervention school in the same district. A framework (Table 1) was presented to the teachers together with appropriate lessons and background notes (See Owens, Reddacliff, Gould & McPhail, (2001) for the framework for Part-Whole Relationships).

Year 2 teachers gave five lessons on part-whole relationships to prepare the students to move on to ten lessons on orientation and motion. These learning experiences emphasised introductory discussions, activity with teacher questioning, and follow-up discussions as a class (see the learning and teaching activity). Teaching occurred over ten weeks and the non-intervention schools followed their existing space program for ten weeks.

Learning experiences on part-whole relationships covered activities like radiating lines from a dot and joining them to make triangles, making shapes with their bodies – with string, with sticks and on a geoboard – cutting up a large triangle into smaller triangles, and covering a 2D shape with the same shape pattern blocks. The learning experiences on orientation and motion are listed above. The interview tasks require students to

1a. explain how a right-angled triangular tile might cover an equilateral tri angle with the base above the vertex,

1b. explain how to move three jigsaw shapes to cover a shape design. The pieces need sliding, turning, and flipping. (new in second year of project),

2. copy an angle made with sticks, copy the angle when shown then covered, and copy when covered and rotated,

3. make triangles with a loop of string, talk about them, and then draw the triangles. One part of the string is held to form one side of the triangle by the interviewing teacher (see figure 1),

4. explain how a net of an open cube is folded to make the cube,
5. predict where faces may be and what they may look like if a square pyramid is placed in different orientations.

Each teacher selected and assessed eight students, two from the top of the class, two from the bottom, and four from the middle (excluding students with extreme giftedness or learning difficulties) prior to the intervention period and afterwards. The researcher assessed the eight students a third time six months after the posttest. There were 73 students assessed in total. In the five comparison schools, a total of 34 students were assessed by a researcher on all three occasions. Chi square analyses compared the number of students in each group who had improved for each task and for those who improved on three or more tasks. The assessed students were given a series of pen and paper tasks *Thinking About 3D Shapes* (Owens, 2001) at the time of the third testing. Confidence intervals of the means of scores were used for comparing the two groups.

Results

Overall, the rate of student progress at the intervention schools was significantly greater than at the non-intervention schools (see table 2). The series of chi-square tests confirmed that on every task (χ^2 ranged from 3.97 to 8.97) and for three or more tasks ($\chi^2_{(1, n=107)} = 5.55, p < .02$) significantly more students from the intervention schools were assessed at a higher level following the intervention compared with students at the non-intervention schools. Eleven percent of students from the intervention schools were assessed at a higher level on all five assessment tasks but none from the non-intervention schools.

Table 2. *Comparison of number of students who improvement on assessment tasks.*

Task	Number (%) who improved with intervention $n = 73$	Number (%) who improved without intervention $n = 34$	χ^2 value	Number (%) who improved with intervention on delayed assessment $n = 37$	Number (%) who improved without intervention on delayed assessment $n = 34$	χ^2 value
1A. Flip triangle	33 (42)	9 (26)	4.48 *	14 (38)	12 (35)	0.05
2. Angle, rotation	43 (59)	13 (38)	3.97 *	17 (46)	18 (53)	0.34
3. Dynamic triangles	42 (58)	9 (26)	8.97 *	20 (54)	17 (50)	0.12
4. Net	44 (60)	12 (35)	5.80*	23 (62)	27 (79)	(2.53)***
5. Faces of pyramid	38 (52)	8 (24)	7.70*	13 (35)	12 (35)	0.00
Three or more tasks	37 (51)	9 (26)	5.55 *	16 (43)	17 (50)	0.33
All tasks	8 (11)	0 (0)	**	1 (3)	4 (12)	

Note. $^*p < 0.05$ *level*; $^{**}p < 0.01$ *** *non-intervention group > intervention group*

After six months delay, the improvement on different tasks varied and the difference between groups was not significant. There are several reasons why this may be the case, among them four seem most reasonable: (a) The overall number of students assessed in the intervention schools was smaller due to an external decision that may have resulted in a non-representative group; (b) In one of these schools there was a noticeable drop in the students who had improved as their next teacher used mainly textbook exercises and several students transferred to other schools; (c) The non-intervention school teachers were keen to be involved in the project and tended to use a great deal of discussion, group work, and hands-on activity already; (d) The researchers assessed the students after six months rather than the teachers.

We thought it would be interesting to compare the results of table 2 with those from one non-intervention school which participated in the intervention after the six-month delay (table 3). Results indicate that students improved their use of strategies in orientation and motion after participating in the lessons. Without intervention there was little gain at the posttest except for the task involving folding a net, which they could remember. By contrast, between 30% and 70% improved on the various tasks after intervention.

Table 3. *Number and percentage of students who improved with non-intervention and then intervention.*

Task	Number (%) who improved without intervention at post assessment $n=7$	Number (%) who improved without intervention at delayed assessment $n=6$	Number (%) who improved with intervention $n=17$
1A Flipping triangle	1 (14)	1 (17)	9 (53)
1B Jigsaw	*Not included*	*Not included*	5 (29)
2Angle, rotation	3 (43)	0 (0)	12 (71)
3 Dynamic triangles	3 (43)	2 (33)	9 (53)
4 Net	2 (29)	5 (83)	8 (47)
5 Faces of pyramid	2 (29)	0 (0)	5 (29)
Three or more tasks	2 (27)	0 (0)	11 (65)
All tasks	0 (0)	0 (0)	3 (18)

Comparison of scores of low attainers on the 3-D test

The three-dimensional test has items similar to those used for spatial abilities on intelligence tests, so this affected the overall results and after the six-month delay, the confidence interval of the mean of scores for the groups overlapped. However, when the students were broken into three groups according to their pre-intervention scores, there was virtually no overlap for those students in the lowest group – intervention had confidence intervals for the mean of 38±3.5 and non-intervention 31±3.5 (see figure 2). This confirms reports by teachers that the weaker students gained considerably from the classroom experiences. The group work, discussion and hands-on experiences encouraged a sense of ownership of their work and helped these students to improve. The program captured the essence of the research especially in developing imagery for (a) recognition of 2D symmetry and 2D and 3D shapes in different orientations, (b) modifying shapes that keep certain properties (dynamic changes), (c) perceiving parts of 3D shapes and (d) imagining 2D nets of 3D shapes.

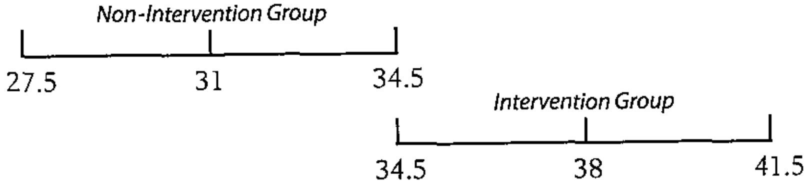


Figure 2. *Performance of low attainers from both groups on three-dimensional shape test.*

Subsequent studies

In the two years (2001 and 2002) there were four more trials in 60 schools with a teacher from each school receiving training and time to act as a school facilitator. Improvements were made in the program in response to my evaluation and teachers' recommendations. These included (a) ordering the lessons and matching them to the strategies; (b) providing videotapes illustrating the strategies and the types of lessons; (c) providing a glossary of terms; (d) revising for teachers the classification of quadrilaterals containing rectangles and rectangles containing squares; and (e) modifying the assessment tasks and lessons to help teachers know earlier and later developments in strategy use (figure 1 shows this development). Another improvement was to give facilitators an initial training day, then they implemented the program and wrote more lessons, followed by a sharing day. This achieved greater ownership and familiarity of the program before they helped other teachers.

Teachers assessed six of their students before and after the program and asked some questions on attitudes. Students showed improvement over the short time of the trials. For example, in one of these trials over two thirds improved on each task of part-whole relationships and over half on each task of orientation and motion. Students who were less confident or knew less about space mathematics were more likely to become confident and know more through participation in the program.

There was overwhelming support by the teachers for the program. Teachers commented in response to a questionnaire that they were more aware of student spatial development and the purpose for space lessons. The teachers were beginning to talk in terms of the framework and were able to recognise the students' improved language and to a lesser extent their imagery.

Observed lessons indicated that teachers developed the lessons drawing on their own ideas. This development beyond following a "recipe" lesson was good. The teachers were often drawn to challenge the students rather than expect final knowledge answers in the activities. As they took students out of their current level of knowledge, so too did the teachers accept going outside their own comfort zone with the lessons. Teachers were encouraging students to develop and discuss their visualising and to recognise movement and patterns. They encouraged more description of shapes. In other words, the focus on the two types of learning approaches (i.e., (a) investigating and visualising, and (b) describing and classifying) were effective in focussing teachers' whole class discussions and their approach to the activities.

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